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On Fictionalism in Aristotle's Philosophy of Mathematics

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On Fictionalism in Aristotle's Philosophy of Mathematics

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On Fictionalism in Aristotle's Philosophy of Mathematics

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The aim of this dissertation is to show that Aristotle's ontology cannot provide a model for mathematics. To show this, I argue that (i) mathematical objects must be seen as fictional entities in the light of Aristotle's metaphysics, and (ii) Aristotle's mathematical fictionalism is not compatible with his metaphysical realism. My interpretation differs from that of other fictionalists in denying this compatibility.

For Aristotle, mathematical objects are "something resulting from abstraction (τὰ ἐξ ἀφαιρέσεως)." For example, geometry investigates a man not *qua* man, but *qua* solid or figure. Traditionally, Aristotle's abstraction has been interpreted as an epistemic process by which a universal concept is obtained from particulars; I rather show his abstraction as a linguistic analysis or conceptual separation by which a certain group of properties are selected: e. g., if a science, X, studies *a qua* triangle, X studies the

properties which belong to *a* in virtue of *a*'s being a triangle and ignores *a*'s other properties.

Aristotle's theory of abstraction implies a mathematical naïve realism, in that mathematical objects are properties of sensible objects. But the difficulty with this mathematical naïve realism is that, since most geometrical objects do not have physical instantiations in the sensible world, things abstracted from sensible objects cannot supply all the necessary objects of mathematics. This is the so-called "precision problem."

In order to solve this problem, Aristotle abandons his mathematical realism and claims that mathematical objects exist in sensibles not as actualities but 'as matter (ὑλικῶς).' This claim entails a mathematical fictionalism in metaphysical terms. Most fictionalist interpretations argue that the fictionality of mathematical objects does not harm the truth of mathematics for Aristotle, insofar as objects' matter is abstracted from sensibles. None of these interpretations, however, is successful in reconciling Aristotle's mathematical fictionalism with his realism.

For Aristotle, sciences are concerned with 'what is (τὸ ὂν)' and not with 'what is not (τὸ μὴ ὂν).' Aristotle's concept of truth rests on a realist correspondence to 'what is (τὸ ὂν)': "what is true is to say of what is that it is or of what is not that it is not." Thus, insofar as mathematical objects are fictional, Aristotle's metaphysics cannot account for the truth of mathematics.

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Abbreviations and Notes on Translations

Works of Aristotle

Analytica Priora	*Apr.*
Analytica Posterior	*Apo.*
Categoriae	*Cat.*
De Interpretatione	*Int.*
De anima	*DA.*
De caelo	*DC.*
De generatione animalium	*GA.*
Ethica Nicomachea	*NE.*
Metaphysica	*Met.*
De Memoria	*DM.*
Physics	*Phys.*
Politica	*Pol.*
Topica	*Top.*

All translations, unless otherwise stated, are mine. I have consulted various existing translations, but focused especially on the volumes of the Clarendon Aristotle Series and the Revised Oxford translation of Aristotle. I have used square brackets, '[…],' to indicate places where certain words have to be supplied.

Introduction

Although there is a fairly general consensus on what Aristotelianism is, there is still an ongoing debate on the nature of his philosophy of mathematics. There are two main reasons for this failure on commentators' part to pin down his views. Despite a number of references to the subject,[1] Aristotle's corpus nowhere features a systematic presentation of his philosophy of math. Another reason may be an ambivalence felt by Aristotle towards the theme. While Aristotle never doubts that mathematics is true in the sense that there are mathematical facts described by the science (or discipline) of mathematics, he declines to identify mathematical objects with anything in the actual world. More puzzlingly, he argues that a mathematical entity cannot exist separated from sensible objects. Mueller describes Aristotle's bind in this respect as resulting from a conflict between a Platonist epistemology of mathematics and non-Platonic ontology.[2] While this attitude makes Aristotle's theories complicated and even dubious, it also makes his position philosophically interesting.

The prevalence of physicalism, in recent years, has led to the inception and

[1] Most but not all of Aristotle's remarks on mathematics can be found in *Mathematics in Aristotle* by Heath (Heath (1949)). Aristotle deals with the problems of philosophy of mathematics mainly in *Met*. XIII and XIV. But a full account of Aristotle's philosophy of mathematics requires consideration of a much broader range of material, such as *Phys.*, III and IV; *DA*., I and III; *APo.,* I; *DC*., II; *Met*., X, etc. In particular, *Phys*., III and IV are essential to any reconstruction of Aristotle's positive theory; *Met*., XIII and XIV focus more on Aristotle's criticism of Platonists' theory of mathematics.

[2] For more detailed discussion of the dilemma which Aristotle proposed to solve regarding mathematical objects, see Mueller (1970) p. 157; Hussey (1992) in Mueller (ed.) (1991) Vol. 24, n. 4, p. 133.

strengthening of several nominalization programs in the philosophy of mathematics.[3] However, as Benaceraff points out, although nominalists can provide themselves with a scientifically acceptable epistemology, they have trouble in providing a model for mathematics.[4] In fact, we see that Aristotle struggled with the same problem: how to account for the truth of mathematics without positing Platonic entities.

Aristotle's discussions of philosophical issues concerning mathematics bear many points of close similarity with modern debates in the philosophy of mathematics. For instance, there are similarities between *Met.,* XIII-XIV and Frege's *The Foundation of Arithmetic*. In XIII-XIV, Aristotle deals with such problems as the ontological status of mathematical objects, the applicability of mathematics, the reducibility of number to units, etc., with wit and sometimes sarcasm, just like Frege.

Nevertheless, the purpose of Aristotle's philosophy of mathematics is not the same as his successor's. While Frege devotes his philosophical and logical work to providing foundations for mathematics, in the belief that that mathematics in his time rested on shaky grounds, Aristotle does not feel the need to provide a philosophical grounding for mathematics. Instead, Aristotle tries to gain credit for his metaphysics by showing that his metaphysics can provide a semantics for mathematics.

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[3] For the development of such recent programs and the current debates, see Irvine (ed.), (1990) pp. xxii and xvii-xxiii.

[4] Benaceraff (1973). Benaceraff diagnoses that the problem in the philosophy of mathematics arises from the conflict between the semantics and the epistemology that tends to go with mathematics. He says that 'two quite distinct kinds of concerns have separately motivated accounts of the nature of mathematical truth: (1) the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language, and (2) the concern that the account of mathematical truth mesh with a reasonable epistemology.' (Benaceraff (1983) p. 661).

The motivation behind Aristotle's development of his theory of mathematics is to criticize metaphysical Platonism.[5] Plato argues for the existence of Forms by relying on differences between objects of sciences and sensible objects; [6] mathematical objects are his favorite examples for revealing such differences, in that they differ by definition from sensible objects in not having sensible properties such as color or weight. Plato argues that, since objects of sciences cannot be sensible objects, they must be Forms. Aristotle's philosophy of mathematics can be seen as his response to such a Platonic argument. If there is any single thesis Aristotle consistently maintains in his theory of mathematics, it is that Platonic Forms are not necessary to explain mathematical truth. To prove this thesis, Aristotle tried to show that mathematical objects are identifiable with some entities existent in his ontology.

I hypothesize that Aristotle makes two different attempts to show the absence of any need to posit Platonic entities to account for mathematical truth.[7] Initially, Aristotle seeks to show that mathematical objects are certain aspects or properties of sensible objects. In order to do this, he introduces a theory of abstraction.[8] Aristotle's abstraction is different from that of the nominalists, which is an epistemic process for obtaining a universal concept from particulars.[9] In my view, Aristotle's abstraction describes a linguistic analysis identifying the primitive subject of a predicate, that is to say, removing

[5] In Chapter One, §1, discuss what metaphysical issues Aristotle aim to resolve in his philosophy of mathematics.
[6] For Plato's arguments from sciences, see Chapter One, §1.
[7] For grounds for this hypothesis, see Chapter Three, §4.
[8] I deal with Aristotle's theory of abstraction in Chapter Two.
[9] For differences between Aristotle's abstraction and and that of the nominalists see Chapter Two, §5.

or abstract other contingent features of a thing to discover which property (or properties) belongs to in its own right.[10] This interpretation is also distinguished from the traditional one according to which abstraction describes a mental operation by which the intellect separates a universal from sensible elements.[11] Aristotle successfully shows how we can conceptually separate a certain group of properties from a sensible object by means of this linguistic abstraction.

Based on this, Aristotle argues that mathematical objects are nothing but properties of sensible objects conceptually separated by abstraction; since such separation by abstraction is only conceptual, mathematical objects do not exist ontologically separated from sensible objects. We may call this position mathematical naïve realism.[12] However, faced with the so-called precision problem, namely that properties of sensible objects do not satisfy the definition of mathematical objects,[13] Aristotle later makes the different claim that mathematical objects are not in sensible objects and exist only as matter. Since this claim is incompatible with mathematical naïve realism, I argue, it is reasonable to suppose that it represents an entirely alternative line of reasoning, marking a break with Aristotle's original position.

Nevertheless, it is not clear what Aristotle meant to say by the claim that mathematicals exist as matter. As the remark is not developed into a complete theory, it leaves open the possibility of various interpretations. We will examine four principal

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[10] For my analysis of Aristotle's abstraction, see Chapter Two, §3.
[11] For my crticisim of the traditional interpretation of Aristotle's Abstraction, see Chapter Two, §5.
[12] In Chapter Three, §1 clarify issues involved with Aristotle's philosophy of mathematics.
[13] For Aristotle's naïve mathematical realism and the precision problem, see Chapter Three, §1.

positions of these.[14]

I begin by considering two so-called fictionalistic positions. One is Lear's view that mathematical objects are constructed out of basic elements such as points, lines, or circles, and that those elements are obtained from sensible objects by abstraction.[15] The other view is that we abstract only pure extension from sensible objects, constructing geometrical objects through our intellect's imposing forms on this extension.[16] This view identifies pure extension with the matter of mathematical objects. Both views can avoid the precision problem, while accommodating Aristotle's thesis that mathematical objects are obtained from sensible objects by abstraction. They both, however, run into the same problem that they turn mathematical objects into fictional entities.

In recent years, there has been a tendency to interpret Aristotle as expressing some kind of fictionalism: For Aristotle, mathematical objects are constructed fictional entities grounded in the actual world. Nonetheless, it is hard to see how mathematical fictionalism can be squared with Aristotle's scientific realism; and given the realism of his theory of truth, mathematics cannot be true if mathematical objects do not exist. This is the only possible consistent fictionalistic interpretation of Aristotle since there is no way that mathematics can be true for him if mathematicals do not exist.[17]

Despite these differences between Aristotle's view and fictionalism, there is one good reason that commentators tend to view Aristotle's view as fictionalism: the claim

[14] I examine these four positions in Chapter Four.
[15] For Lear's view and my criticism, see Chapter Four, §1.
[16] I review this constructivistic interpretation in Chapter Four, §2.
[17] For my criticism of the fictionalistic approach to Aristotle's view of mathematics, see Chapter Four, §1.

that mathematical objects are not in sensible objects and exist only in matter makes it difficult to identify mathematical objects as any kind of actual entities in Aristotle's metaphysics.[18] This issue, though, may be more complicated than it first appears; 'being' has many different senses in Aristotle. Aristotle's ontological inventory features not only all items belonging to each category as beings, but also potential beings as these are contrasted with actual beings.[19] In addition, Aristotle frequently uses the notion of 'matter' to mean 'potentiality,' e.g., the matter of a statue is the potentiality of the statue. Thus, there would seem to be a way to make mathematical objects something existent by interpreting the matter of mathematical objects, i.e., pure extension, as the potentiality of mathematical objects.[20]

A difficulty with this interpretation is, though, that Aristotle uses the term 'potentiality' homonymously, so that not every kind of potentiality can be seen as a mode of existence.[21] Only when the matter of mathematical objects can be called potentiality in the sense of another mode of existence, will there be a ground to say that mathematical objects *exist* in Aristotle's ontology. However, pure extension, the matter of mathematical objects, differs from those potentialities which are considered to involve another mode of existence.[22] First, unlike, for instance, an incomplete substance such as a boy, extension *qua* potentiality does not have the internal causal power to actualize itself or something

[18] For the idealistic aspect of Aristotle's philosophy of mathematics, see Chapter Three, §5.
[19] For discussion of potentiality as another mode of existence, see Chapter Four, §3, 3,3.
[20] In Chapter Three, §3, I discuss the view that mathematical objects can be assimilated to potential beings in Aristotle's ontology.
[21] For Aristotle's homonymous uses of the term, 'potentiality,' see Chapter Four, §3, 3.2.
[22] For differences between the matter of mathematical matter and other potential beings involving a mode of existence, see Chapter Four, §3, 3.3.

else into its actuality. Secondly, while Aristotle maintains that actuality is prior to its potentiality in existence in the sense that potentiality's existence depends on actuality's, the existence of pure extension is prior to that of any of its possible actualizations, i.e., geometrical figures. Moreover, the fact that geometrical objects are not actualized in the sensible world makes it doubtful whether geometrical objects exist in any form of actuality at all. For Aristotle, something which is only in potentiality and never actualized, (e.g., infinity) is considered not as existent but rather as a kind of non-being.[23]

Hintikka claims, though, that there is an actuality of each geometrical object; geometrical objects are actualized in the intellect, and such mental actualization hardly differs for Aristotle from physical actualization in the external world.[24] This argument is based on Aristotle's claim of an identity between the thinking intellect and objects of thinking. Since, when the intellect thinks some object, *a*, it becomes identical with *a*, Hintikka infers that when *a* is thought by the intellect, *a* is actualized therein. However, when Aristotle makes the claim of an identity between the subject and objects of thought, he uses the term 'thinking' in a specific sense: by 'thinking' in this context he means knowing. When 'thinking' is used in this sense, the objects of thinking are restricted to the forms of existing things in the external world. And since mathematical objects do not exist in the external world, they cannot be objects of thinking, either.

Due to the realism of his theory of truth, for Aristotle, mathematics can be true

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[23] For the ontological status of non-actualizable potentiality in Aristotle's metaphysics, see Chapter Four, §3, 3.4.

[24] I discuss Hintikka's view in Chapter Four, §4.

only if mathematical objects exist.[25] Nevertheless, neither Aristotle nor his commentators have shown that mathematical objects can be identified with some entities in his ontology. In this sense, Aristotle's metaphysics cannot be a model for mathematics. The conclusion of this work is thus that Aristotle was not successful in showing how mathematics can be true within his metaphysics.

[25] For the necessity of the existence of mathematical objects for an account of the truth of mathematics in Aristotle's metaphysics, see Chapter Three, §1.

Chapter One

Aristotle's Scientific Realism and Mathematics

1. The Arguments from Sciences

Among Plato's many arguments for the existence of Forms in Aristotle's works, there are the so called 'arguments from sciences (AFS).'[26] These arguments posit that there will be Forms of all things of which there are sciences.[27] Given that one of the motivations of Platonism is to explain how scientific knowledge (ἐπιστήμη) is possible, the AFS can be understood as critical in the continuation of a Platonic heritage beyond the ideas of Plato himself.

Unfortunately, despite their importance, the arguments from sciences are not stated in their entirety in Aristotle's surviving works. This makes it an open question whether the arguments were taken directly from Plato or rather reframed by Aristotle himself. Nevertheless, the fact that Aristotle refers to the arguments without further citational specification is enough to suggest that the arguments would have been familiar to members of the Academy.

[26] Other Platonists' arguments for the existence of Forms can be found at *Met*., I, 9, 991b20-992a10; VII, 8, 1033b20-1034a9; XI, 2, 1060a2-26; *NE.,* I, 6, 1096a17-1096b6, etc.

[27] *Met.,* XIII, 4, 1079a7-8. Aristotle also mentions the arguments of the sciences at *Ibid.,* I, 9, 990b11-12. 1079a7-1079b3 is almost verbally identical to 990b11-991a9.

Despite this central failure in transmission, we do owe one version of the complete form of the AFS to Alexander of Aphrodisias,[28] who supposes that they are presented in full in a lost work of Aristotle's called *On Forms*.[29] The arguments are taken to go as follows:[30]

> (1) If every science does its work with reference to a single identical thing, and not to any other particulars, there must be, corresponding to each science, something other than sensible things, which is eternal and is the pattern of the particulars in each field of the science in question. Now that is just what the Form is.
>
> (2) The things of which there are sciences must exist; now the sciences are concerned with things other than particular things; for the latter are indefinite and indeterminate, while the objects of the sciences are determinate; therefore there are things other than the particulars and these are the Forms.
>
> (3) If medicine is the science not of this particular instance of health, but just of health, there must be such a thing as health-itself, and if geometry is

[28] Alexander was the most celebrated Ancient Greek commentator on Aristotle. His commentaries on *Prior Analytics* (Book I), *Topics, Meteorology, Sense and Sensibilia,* and *Metaphysics* (Book I-V and a summary of his commentary on the rest of *Metaphysics*) still survive and significantly shaped the Arab reception of Aristotle, some in Arabic translation.

[29] Alexander of Aphrodisias, *On Aristotle Metaphysics,* 79.5-15 (see Alexamder (1989) pp. 115-116). Some of the purported arguments may also be found in fragments 3 and 4 of *On the Forms* in Ross (1952). For the disussion of AFS, see Cherniss (1944); Owen (1957); Annas (1975a).

[30] Trans., J. J. Cleary (see Cleary (1987) p. 97).

knowledge not of this equal and this commensurate, but of what is just equal and what is just commensurate, there must be an equal-itself and a commensurate-itself, and these are the Forms.

Whether or not these arguments were presented by Plato himself or reconstructed later by someone else, these Platonic claims about sensible particulars and the Forms are found in many places in the *Dialogues*.[31] In *Phaedo,* 74a-c, for instance, Plato argues that equal things are different from the Equal itself; since the former are deficient—things are sometimes equal but sometimes not—only the latter may stand an object of knowledge.[32] Similarly, (1) argues that a science does not deal with particular *F*-things but a single item, *F* itself, e.g., geometry does not study triangular things such as a bronze isosceles triangle but the triangle itself. Since F-things are different from *F*-itself, *F*-things cannot be objects of a science; (3) may be regarded as complementing (1) by examples.

Meanwhile, according to the second argument, while the objects of a science are definite, particulars are indefinite, so that the objects of a science must be other things than particulars. We can find the reason that Plato thinks that particulars are indefinite in

[31] Ross identifies the arguments from the sciences with arguments similar to those presented at *Republic*, 479a-480a, and *Timaeus*, 51d-52a. On the basis of these passages, Ross reconstructs the AFS as follows: "If knowledge exists, there must exist an unchangeable object of knowledge. Knowledge does exist. Therefore there exists an unchangeable object. Sensible objects are changeable. Therefore there exist non-sensible realities." Ross (1924) *Vol. I*, p.193.

[32] This deficiency or imperfection of sensible particulars is an important issue in Aristotle's philosophy of mathematics, which has been parsed as 'the precision problem', namely that sensible objects do not perfectly instantiate mathematical objects. For a discussion of the precision problem, see Chapter Two, §3.

the *Dialogues*. Plato's argument here is that sensible particulars are in constant change and so never in the same state,[33] and any account given of thing in such a state will similarily lack accuracy and consistency.[34] Futher, according to Aristotle, at the very beginning, for Plato, the search for knowledge led to definitions, but Plato thought that defintions could not be found in things in constant change. Aristotle reports that:

> Socrates, being busy himself about ethical matters, but nothing about the whole nature and seeking the universal in these [ethical matters], gave his attention for the first time to definitions. Plato, accepting this through such as this [procedure], supposed that this comes to be about different things, and not about sensibles; because it is impossible that the common definition be about any of the sensibles, for they are always changing.
>
> Σωκράτους δὲ περὶ μὲν τὰ ἠθικὰ πραγματευομένου περὶ δὲ τῆς ὅλης φύσεως οὐθέν, ἐν μέντοι τούτοις τὸ καθόλου ζητοῦντος καὶ περὶ ὁρισμῶν ἐπιστήσαντος πρώτου τὴν διάνοιαν, ἐκεῖνον ἀποδεξάμενος διὰ τὸ τοιοῦτον ὑπέλαβεν ὡς περὶ ἑτέρων τοῦτο γιγνόμενον καὶ οὐ τῶν αἰσθητῶν: ἀδύνατον γὰρ εἶναι τὸν κοινὸν ὅρον τῶν αἰσθητῶν τινός, ἀεί γε μεταβαλλόντων.[35]

To clarify the issues on which Aristotle begins to diverge from Plato, we should consider the arguments' main theses again. Here we will frame the arguments even more succinctly:

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[33] *Phaedo*, 78e.

[34] *Timaeus*, 29b3-5. Plato draws a distinction between *wha always is and never become* (τί τὸ ὄν ἀεί γένεσιν δὲ οὐκ ἔχον) and *what becomes and never is* (τί τὸ γιγνόμενον μὲν ἀεί ὄν δέ οὐδέποτε) (*Timaeus*, 27d5-28a), and connects each with its epistemological counterpart: the former is grasped by νόησις and the latter by δόξα (*Ibid.*, 28a1-4).

[35] *Met.*, I, 6, 987b1-7.

1. Sciences are of something existent (τὸ ὄν)

2. So, there must exist objects of sciences.

3. Sensible particular things differ from the objects of sciences; the former are indefinite, the latter definite.

4. There must be objects of sciences other than sensible particulars.

5. Those things are Platonic forms.

An underlying assumption of these claims is that sciences exist as bodies of knowledge.[36] Neither Aristotle nor Plato addresses the question of whether there *is* any knowledge at all, one familiar to modern epistemology. They rather accept the existence of knowledge as a given fact. The Academy had from the outset taught a great variety of disciplines, divided into the two categories of Physics and Mathematics, the former including natural sciences such as zoology and botany and the latter incorporating arithmetic, geometry, astronomy and harmony.[37] Their interest, then, lies in the nature of the objects of knowledge and the conditions that make knowledge possible.[38] Further, it

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[36] Here 'science' is the translation of *'ἐπιστήμη'*. Depending on the context, ἐπιστήμη could mean 'knowledge' as opposed to δόξα, (*Rep*. V, 476a-480a; VI, 509d-511e; VII, 514a-521b), 'theoretical knowledge' as opposed to ποιητική (*NE*, VI, 5), or 'an organized body of knowledge' (*Met.,* VI, 1).
*All translations, unless otherwise stated, are mine. I have used square brackets, '[…],' to indicate places where certain words have to be supplied.

[37] Lasserre (1964) p. 15. But mathematics was not yet rigorously systematized in Plato's time. The Academy seems to have used the *Elements* of Hippocrates of Chios, which does not survive (see Maziarz (1968) p.98).

[38] For Aristotle's criticism of skeptics and those who assert that all knowledge requires demonstration, refer to *Apo.*, I, 2, 72b5-73a20.

is Aristotle who develops the concept of a science as an organized body of rational knowledge with its own proper object.[39]

Aristotle also shares with Plato the Parmenidean legacy that knowledge is of beings (τὸ ὄν). For Aristotle, a demonstrative science or demonstration (ἀπόδειξις) must be about something (περί τινῶν)[40] and 'each science marks off a certain class of things for itself and busies itself about this as about something that exists and is (ἑκάστη γὰρ τούτων περιγραψαμένη τι γένος αὑτῇ περὶ τοῦτο πραγματεύεται ὡς ὑπάρχον καὶ ὄν).[41] So, as Apostle points out, for Aristotle, "sciences are concerned with what exists and not with what does not exist."[42]

Moreover, since Aristotle agrees with Plato that particulars cannot be the objects of knowledge,[43] he also argues that something other than particulars should be the objects of sciences, identifying these with universals. He argues that, if there are no universals, there will be no middle terms, and hence no demonstrations; but, since there are actually sound demonstrations, there must be universals. The middle term can be said to be a universal because it must be distributed in at least more than one item in a valid inference; because a demonstration requires true pairs of universal affirmative propositions like AaB & BaC, where B is the common term which links both items A and

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[39] For Aristotle, scientific knowledge (ἐπιστήμη) is the knowledge of a cause (αἰτία) (*APo.*, I, 10, 76a25-30), while opinion (δόξα) is about the contingent (συμβεβηκός) (*Ibid.*, 33, 88b30-89a5).

[40] *Republic*, V, 476e-477a; *Met.,* 997a5-6; *Met.*, XI, 7, 1064a2-3; *APo.,* I, 11, 77a6-77a9.

[41] *Met.,* XI, 7, 1064a2-3.

[42] Apostle (1952) p.11. Cf. *Apo*, I, 10, 76b17-23.

[43] Knowledge is always about universals. See, *Met*. VII. 13.

C.[44] The universal is "what by nature is predicated of plural things (ὃ ἐπὶ πλειόνων πέφυκε κατηγορεῖσθαι),"[45] such that it can be true of many.

Aristotle's argument for the existence of universals recalls Plato's argument for the existence of Forms. Just as Plato argues for the existence of Forms on the basis of the existence of knowledge, Aristotle argues on the basis of the truth of a science. Why does Aristotle make such an effort to prove the existence of universals? We should remember here that, for Aristotle, the sciences deal with universals. Thus, in order to maintain his scientific realism, it is necessary for him to show that universals exist. Aristotle's endeavor to prove the existence of universals thus represents another facet of his strong scientific realism.

This parallel between Plato's and Aristotle's constitution of scientific knowledge suggests that Aristotle agrees with his forebear on all the theses 1 through 4. What Aristotle denies is only that 5 follows from them. He says on this topic, "Thus, it is not necessary that there is to be a form or some one thing apart from the many, even if there are to be demonstrations (εἴδη μὲν οὖν εἶναι ἢ ἕν τι παρὰ τὰ πολλὰ οὐκ

[44] "If there is to be demonstration, however, it is necessary that one thing is true of many; for if this is not the case, there will be no universal; and if there is no universal, there will be no middle; thus, there will be no demonstration, either. Therefore, there must be some thing and the same thing, non-homonymous, [holding] of many things (εἰ ἀπόδειξις ἔσται, εἶναι μέντοι ἓν κατὰ πολλῶν ἀληθὲς εἰπεῖν ἀνάγκη: οὐ γαρ ἔσται τὸ καθόλου, ἀν μὴ τοῦτο ᾖ: ἐὰν δὲ τὸ καθόλου μὴ ᾖ, τὸ μέσον οὐκ ἔσται, ὥστ' οὐδ' ἀπόδειξις. δεῖ ἄρα τι ἓν καὶ τὸ αὐτὸ ἐπὶ πλειόνων εἶναι μὴ ὁμώνυμον (*APo.,* I, 11, 77a6-77a9))."

[45] *Int.,* 7, 17a39-40. See also *Met.,* 13,1038b11. Cf. *Apr.,* I, 1, 24a17-18; 24b28-30.

ἀνάγκη, εἰ ἀπόδειξις ἔσται)."[46] Although Aristotle assumes universals as the objects of science, Aristotle's universals are different from Platonic Forms, in that they are not separated from particulars but inhere in individuals. We should note that an important features of a Form is its separateness from particulars; for Plato, each Form is an 'αὑτὸ καθ' αὑτό' being, namely, it exists in its own right[47]. On account, further, of the inseparability of universals from individual objects, Aristotle's scientific realism emerges as a form of anti-Platonism, even as both Aristotle and Plato posit some entity entities other than individual particulars as the objects of a science.

This analysis reveals two features of Aristotle's philosophy of the sciences: its realism and anti-Platonism. Insofar as Aristotle takes issue with the idea that the objects of sciences are necessarily separated from particulars, he is an anti-Platonist. Nevertheless, his position cannot simply be assimilated to nominalism; since he accepts the existence of scientific objects other than particulars, namely, universals as having real existence.[48]

2. Aristotle's Mathematical Realism

Aristotle's philosophy of mathematics reflects his view of science in general, embodying his conception of mathematics as the prime example of a science. On the one hand, as

[46] *Apo.,* I, 11, 77a5-7
[47] *Parmenides.,* 130b-c.
[48] Barnes labels him as a weak Platonist in this regard. See Barnes (1985) p. 98.

an anti-Platonist, Aristotle objects to the idea that mathematical objects exist independently in ontological terms of sensible particular substances.[49] Since only substance can exist by itself, this means that Aristotle must deny that mathematical objects are substances.[50] He is thus led to argue no separation between mathematical objects and the sensible particulars that embody them.[51] But that does not mean that mathematicals do not exist, any more than the inseparability of other categorial beings from individual substances entails in turn these beings' non-existence. Aristotle says:

> Thus since it is true to say without qualification that not only things which are separable but also things which are inseparable exist (for instance that moving things exist), it is true also to say without qualification that the objects of mathematics exist and such things as they [mathematicians] are talking about.
>
> ὥστ᾽ ἐπεὶ ἁπλῶς λέγειν ἀληθὲς μὴ μόνον τὰ χωριστὰ εἶναι ἀλλὰ καὶ τὰ μὴ χωριστά (οἷον κινούμενα εἶναι), καὶ τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν, καὶ τοιαῦτά γε οἷα λέγουσιν. [52]

[49] For the ontological dependency of mathematical objects on sensible individuals, see *Met.*, XIII, 2, 1077a16; 1077a26; 1077b1.

[50] Substances are always ontologically prior to the other kinds of beings. For the ontological dependency of other categorical beings on substance, see *Cat.,* 5, 2a34.

[51] For Aristotle's claim of the inseparability of mathematical objects from sensible things, see *Met.,* XIII, 2, 1076b12; 1076b36; 1077a2; 1077a9. For instance, lines are not substances (*Ibid.,* 1077a32); and number cannot exist separately from sensibles (*Ibid.,* 8, 1083b10; 1083b31; 9, 1085a26).

[52] *Ibid.,* 1077b31-34.

Aristotle here argues that not only separable things but also inseparable things can be said to exist. Since, for Aristotle, the categories are ways of being and other categories of beings are inseparable from substances, he can further claim that mathematical objects exist without being concerned with their inseparability. Aristotle confirms the existence of mathematical objects in several passages. For example, at the opening of *Met.*, XIII, his discussion of the ontological status of mathematicals makes it clear that the subject of the chapter is not whether mathematicals exist but rather how: "the subject of our discussion will be not about whether they exist but how they exist (ἡ ἀμφισβήτησις ἡμῖν ἔσται οὐ περὶ τοῦ εἶναι ἀλλὰ περὶ τοῦ τρόπου)."[53] Thus, for Aristotle, the existence of mathematicals is not in question; the only ontological problem concerning mathematical objects is their mode of existence.[54] Aristotle diverges from mathematical Platonism not on the ground of the non-existence of mathematical objects but by means of a denial of their self-subsistence.

3. Aristotle's Criticism of Mathematical Platonism

Aristotle's supposed refutation of mathematical Platonism is especially relevant to the AFS; this is not only because the AFS directly imply mathematical Platonism as their corollary but also because both Plato and Aristotle regard mathematics as the best case

[53] *Met.*, XIII, 1, 1076a36-37.

[54] Barnes argues the same point concerning numbers. He says, "For if numbers do not exist at all, then arithmetic studies nothing, i.e., there is no such thing as arithmetic. But that is absurd: it is a datum that arithmetic exists." See Barnes (1985) p. 102.

supporting the existence of scientific objects other than particulars. It seems, then, that Aristotle refutes mathematical Platonism in order to refute the ASF itself. In this sense, *Met.*, Book XIII and XIV, where Aristotle attacks mathematical Platonism, could be seen as his response to AFS.[55] In fact, *Met*., XIII-XIV is the only place in which Aristotle directly deals with these arguments.[56]

It is an interesting issue whether Aristotle's criticism of mathematical Platonism in *Met*., XIII-XIV, is successful. Rather than considering the issue closely, the focus of this dissertation lies on whether Aristotle provides an alternative theory to mathematical Platonism, to give an account of the truth of mathematics. While fascinating, the earlier question is perhaps too large to form the subject of any single dissertation, not least because Plato's own early theory differs markedly from his development of it, even within *Dialogues*. The matter is further complicated by the admission of Plato's so-called unwritten doctrine, and by the departures from Plato's view introduced by his successors. In consequence of this last point, it is difficult to know exactly which positions Aristotle means to place into doubt on different occasions, as in this indicative instance:

> But it is not possible for such a kind of things to exist in separation either; for if there are to be solids besides perceptible solids, separated from them, distinct and prior to perceptible solids, it is obvious that there must be distinct and separate planes besides [perceptible] planes, and the same argument applies to points and lines as well. But if so, there will again be distinct and separate planes and lines and points besides those of the

[55] Annas agrees with this point. For the relationship between AFS and Aristotle's criticism of mathematical Platonism, see Annas (1976) p.23 and 142.

[56] Aristotle seems specifically to allude to AFS at *Met.,* XIII, 4, 1079a7-8. His counter arguments against AFS are found at *Ibid.,* 2, 1076b39-1077a14.

mathematical solids (for the uncompounded is prior to the compounded and if there are non-perceptible ones, by the same argument there are planes existing by themselves prior to those in the unchanging solids. So these planes and lines and solids are distinct from those that belong together with the separate solids; the latter belong together with the mathematical solids, while the former are prior to the mathematical solids.) Then again there will be lines and points in the prior lines; there will be distinct prior points, though there are no more prior to them. The piling up becomes absurd: we get one set of solids over and above perceptible planes (those besides perceptible planes, those in the mathematical solids, and those besides the latter), four sets of lines, and five sets of points, which of them will be the object of the mathematical branches of knowledge? Not the planes and lines and points in the unchanging solids, for knowledge always deals with what is prior.

ἀλλὰ μὴν οὐδὲ κεχωρισμένας γ᾽ εἶναι φύσεις τοιαύτας δυνατόν. εἰ γὰρ ἔσται στερεὰ παρὰ τὰ αἰσθητὰ κεχωρισμένα τούτων ἕτερα καὶ πρότερα τῶν αἰσθητῶν, δῆλον ὅτι καὶ παρὰ τὰ ἐπίπεδα ἕτερα ἀναγκαῖον εἶναι ἐπίπεδα κεχωρισμένα καὶ στιγμὰς καὶ γραμμάς (τοῦ γὰρ αὐτοῦ λόγου): εἰ δὲ ταῦτα, πάλιν παρὰ τὰ τοῦ στερεοῦ τοῦ μαθηματικοῦ ἐπίπεδα καὶ γραμμὰς καὶ στιγμὰς ἕτερα κεχωρισμένα (πρότερα γὰρ τῶν συγκειμένων ἐστὶ τὰ ἀσύνθετα: καὶ εἴπερ τῶν αἰσθητῶν πρότερα σώματα μὴ αἰσθητά, τῷ αὐτῷ λόγῳ καὶ τῶν ἐπιπέδων τῶν ἐν τοῖς ἀκινήτοις στερεοῖς τὰ αὐτὰ καθ᾽ αὑτά, ὥστε ἕτερα ταῦτα ἐπίπεδα καὶ γραμμαὶ τῶν ἅμα τοῖς στερεοῖς τοῖς κεχωρισμένοις: τὰ μὲν γὰρ ἅμα τοῖς μαθηματικοῖς στερεοῖς τὰ δὲ πρότερα τῶν μαθηματικῶν στερεῶν). πάλιν τοίνυν τούτων τῶν ἐπιπέδων ἔσονται γραμμαί, ὧν πρότερον δεήσει ἑτέρας γραμμὰς καὶ στιγμὰς εἶναι διὰ τὸν αὐτὸν λόγον: καὶ τούτων <τῶν> ἐκ ταῖς προτέραις γραμμαῖς ἑτέρας προτέρας στιγμάς, ὧν οὐκέτι πρότεραι ἕτεραι. ἄτοπός τε δὴ γίγνεται ἡ σώρευσις (συμβαίνει γὰρ στερεὰ μὲν μοναχὰ παρὰ τὰ αἰσθητά, ἐπίπεδα δὲ τριττὰ παρὰ τὰ αἰσθητά τά τε παρὰ τὰ αἰσθητὰ καὶ τὰ ἐν τοῖς μαθηματικοῖς στερεοῖς καὶ <τὰ> παρὰ τὰ ἐν τούτοισγραμμαὶ δὲ τετραξαί, στιγμαὶ δὲ πενταξαί: ὥστε περὶ ποῖα αἱ ἐπιστῆμαι ἔσονται αἱ μαθηματικαὶ τούτων; οὐ γὰρ δὴ περὶ τὰ ἐν τῷ στερεῷ τῷ ἀκινήτῳ ἐπίπεδα καὶ γραμμὰς καὶ στιγμάς: ἀεὶ γὰρ περὶ τὰ πρότερα ἡ ἐπιστήμη).[57]

The main point of the argument is that, once we posit ideal geometrical objects as ontologically separate from perceptible ones, we are led into an absurd multiple reduplication of such ideal objects. For example, suppose that there are the physical solid and its corresponding ideal solid. Then, there should be also an ideal plane, line, and point corresponding to the physical plane, line, and point, respectively. But since the ideal solid is composed of planes, there should be ideal planes of the ideal solid. So, we come to have other ideal planes besides the ideal plane we posited along with the ideal solid at the beginning. But since each plane is composed of lines, we will also have posited the ideal line at the beginning, and lines of which the ideal plane is composed, and lines of which planes of the ideal solid are composed. We can apply the same argument to lines and points as well. As a result, we come to have two sets of solids: perceptible and ideal solids; three sets of planes: perceptible, ideal planes and planes as the components of the ideal solid; and by the same logic, four sets of lines and five sets of points.[58]

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[57] *Ibid.,* 1076b11-1076b36. The argument at *Ibid.,* III, 2, 997b33-998a6 is almost the same, but a little longer.

[58] Aristotle argues that this also holds of numbers: "The same argument [applies] also to numbers: for besides each set of points there will be distinct units, and also besides each group of things, besides perceptible things and again besides intelligible things. Thus, there will be [many] kinds of numbers (ὁ δ' αὐτὸς λόγος καὶ περὶ τῶν ἀριθμῶν: παρ' ἑκάστας γὰρ τὰς στιγμὰς ἕτεραι ἔσονται μονάδες, καὶ παρ' ἕκαστα τὰ ὄντα, <τὰ> αἰσθητά, εἶτα

Apart from commentators' disagreement on subtle points of this argument, it would seem that Aristotle's correctness in these claims depends on what theory he is attacking. For example, since Plato in the middle period identifies mathematicals with the *intermediates* rather than Forms, the middle Plato would argue that both the ideal plane and the planes as components of the ideal solid are intermediates; since the intermediates can be plural, it is not problematic that there are plural numbers of the same kind of geometrical objects.[59] The middle Plato may be not thus vulnerable to the reduplication problem. Likewise, the problem need not trouble Speusippus either, because he rejects the existence of Forms, while maintaining the existence of ideal numbers.[60] Since only

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τὰ νοητά, ὥστ᾽ ἔσται γένη τῶν μαθηματικῶν ἀριθμῶν) (*Ibid.,* XIII, 3,1076b35-1076b39)."

[59] The intermediates differ from sensible things in being eternal and unchangeable, but, unlike Forms, the intermediates do not have to be unique; that is, there can be many of the same type. Aristotle says, "besides perceptible things and Forms, he says, there are the objects of mathematics between these [two], differing from perceptible things in being eternal and unchangeable, from Forms in that there are many alike, whereas the Form itself is in each case unique (ἔτι δὲ παρὰ τὰ αἰσθητὰ καὶ τὰ εἴδη τὰ μαθηματικὰ τῶν πραγμάτων εἶναί φησι μεταξύ, διαφέροντα τῶν μὲν αἰσθητῶν τῷ ἀΐδια καὶ ἀκίνητα εἶναι, τῶν δ᾽ εἰδῶν τῷ τὰ μὲν πόλλ᾽ ἄττα ὅμοια εἶναι τὸ δὲ εἶδος αὐτὸ ἓν ἕκαστον μόνον) (*Met.,* I, 6, 987b14-18) ." But it is also controversial whether Plato actually did believe in the intermediates in this sense. According to Aristotle, the notion of the intermediates is introduced as the solution to the uniqueness problem, viz. that while each Form is unique, mathematical objects are supposed to be plural, i.e., a geometrical theorem mentions *two* intersecting circles. But the *Dialogues* do not seem to have noticed this problem nor to be impelled to find a solution to it. The most explicit argument for the intermediates can be found in *Republic,* VI, 509d-511a, and *Philebus,* 56c-59d, but these passages do not throw up the uniqueness problem. For this issue, see Annas (1975a) pp.19-21.

[60] fr. 42a-e. For the testimonia and fragments of Speusippus, refer to Parente (1980) and Leonardo Tarán, (1981) *Speusippus of Athens: A Critical Study with a Collection of the Related Texts and Commentary,* Leiden: E.J. Brill.

Xenocrates retains Forms[61] and identifies them with corresponding mathematical objects,[62] it would seem that only Xenocrates is fully liable to the criticism that Aristotle mounts in this passage.

Nevertheless, some of Aristotle's anti-Platonic arguments deserve attention insofar as they can help us reconstruct his positive view of mathematics and his motivation in introducing abstraction into his theory of mathematics. The following objection, for instance, reveals Aristotle's view of astronomy, which he regards as a branch of mathematics (we will also see that this view gives rise to a problem which requires the theory of abstraction for its solution):

> For the objects of astronomy will exist apart from sensible objects just as the objects of geometry; but how can a heaven and its parts, or anything else exist with movement [apart from the sensible heaven]? And similarly with the objects of optics and harmonics; for there will be utterance and seeing apart from sensible and particular objects. Therefore, it is obvious that the other senses and objects of perception [will exist separately, too]...But if this is the case, there will be [separate] animals too, if there are [separate] senses.
>
> περὶ ἃ γὰρ ἡ ἀστρολογία ἐστίν, ὁμοίως ἔσται παρὰ τὰ αἰσθητὰ καὶ περὶ ἃ ἡ γεωμετρία: εἶναι δ᾽ οὐρανὸν καὶ τὰ μόρια αὐτοῦ πῶς δυνατόν, ἢ ἄλλο ὁτιοῦν ἔχον κίνησιν; ὁμοίως δὲ καὶ τὰ ὀπτικὰ καὶ τὰ ἁρμονικά: ἔσται γὰρ φωνή τε καὶ ὄψις παρὰ τὰ αἰσθητὰ καὶ τὰ καθ᾽ ἕκαστα, ὥστε δῆλον ὅτι καὶ αἱ ἄλλαι αἰσθήσεις καὶ τὰ ἄλλα

[61] fr. 30. For the fragments of Xenocrates, refer to Margherita Isnardi Parente (1982) *Frammenti; Senocrate-Ermodoro ; Edizione, traduzione e commento*, Napoli: Bibliopolis.

[62] "There are Forms of geometrical objects and numbers and mathematics deals with them (fr. 34)."

αἰσθητα...εἰ δὲ ταῦτα, καὶ ζῷα ἔσονται, εἴπερ καὶ αἰσθήσεις.[63]

Aristotle takes astronomy to be a counterexample to AFS. Astronomy is a science lying on the borderline between physics and pure mathematics. On the one hand, it involves the study of physical objects, while on the other, it was treated as a branch of mathematics by the ancient Greeks. At first glance, the argument seems to appeal to our common sense that the objects of astronomy are sensible objects, stars in the heavens; there is no other separate, ideal heaven;[64] there are not ideal sounds as the objects of harmony separated from perceptual sounds, etc. But we should remember that from the beginning Platonism does not appeal to our common sense. On the contrary, it consistently devalues it: opinion (δόξα), which is concerned with the world of physical objects, blinds us to the truth. Thus, the fact that AFS implies something incompatible with our common sense beliefs cannot be a ground for rejecting mathematical Platonism—until we provide a better reason to rely on those beliefs.

But, in the case of astronomy, there is a better reason to think that its objects are sensible. One of the premises of AFS is that sensible objects are deficient as the objects of a science, in that they are in constant change and consequently never in the same state. But it was commonly believed among ancient Greeks that the movements and structures of the heavenly bodies were perpetual, regular and complete. If the movement and

[63] *Met.,* XIII, 2, 1077a1-9.

[64] Aristotle believes that there is only one heaven (*Met.*, XII, 8, 1074a31-33).

structure of the heavenly body are perpetually uniform and regular, there is no reason to posit separate entities as the objects of astronomy other than the visible heavenly bodies. And that suffices to show that AFS is not universally true; namely, it is not always the case that the objects of a science must be separated from physical objects.

Aristotle also argues the same point with optics and harmonics. But, obviously, it does not follow from the fact that the objects of astronomy are not distinct from visible stars that the objects of other sciences are likewise not distict from sensibles; it could be that astronomy here represents an exceptional case. Were this so, AFS would be still valid for a sub-class of all those sensible objects that exhibit no regularity.

In order to appreciate the force of the argument, it is important to grasp the polemical character of Aristotle's claims in *Met.,* XIII-XIV: he focuses primarily on revealing the absurdity of his opponents' positions rather than developing his own, and seeks mostly to persuade opponents of his view rather than to prove it, appealing to plausibility or likelihood rather than any secure demonstration. This polemical characteristic especially comes out in his frequent employment of *reductio ad absurdum* as a purported method of proof. He accepts opponents' theses as the premises of his argument insofar as they serve his purpose even when he likely considers them false.

As suggested earlier, harmony and optics were regarded by Aristotle's contemporaries as the sibling sciences of astronomy, insofar as they are all branches of mathematics.[65] Thus, the Greeks would be inclined to think that they shared common characteristics with astronomy, meaning that Aristotle's assimilation of some properties

[65] *Republic*, VII, 530d.

of astronomy to those of harmonics and optics was likely to dispose his interlocutors to agreement. At the same time, the persuasiveness of Aristotle's argument relies on the assumption that the movements of heavenly bodies are perfect enough to satisfy the criteria for objects of a science. But the following passage of Plato, however, denies just such an assumption:

> But as for the ratio of night to day, of day and night to a month, of a month to a year, or of the other stars to these and one another, don't you think he'll regard as a strange person one who believes that they are always the same and never deviate at all, which possess bodies and are visible objects, and seeks to grasp the truth out of them in some way?
>
> τὴν δὲ νυκτὸς πρὸς ἡμέραν συμμετρίαν καὶ τούτων πρὸς μῆνα καὶ μηνὸς πρὸς ἐνιαυτὸν καὶ τῶν ἄλλων ἄστρων πρός τε ταῦτα καὶ πρὸς ἄλληλα, οὐκ ἄτοπον, οἴει, ἡγήσεται τὸν νομίζοντα γίγνεσθαί τε ταῦτα ἀεὶ ὡσαύτως καὶ οὐδαμῇ οὐδὲν παραλλάττειν, σῶμά τε ἔχοντα καὶ ὁρώμενα, καὶ ζητεῖν παντὶ τρόπῳ τὴν ἀλήθειαν αὐτῶν λαβεῖν;[66]

Plato here argues that the motions of heavenly bodies do not meet the criteria for the objects of a science because they are not always in the same state; thus they are not the objects of astronomy. Since Plato thinks that the visible motions of stars are not the objects of astronomy, he has every reason to apply AFS to the case of astronomy, too. Thus he argues:

[66] *Republic,* VII, 530a7-b4.

> We should consider the decorations in the sky to be the most beautiful and most exact among such [visible] things, seeing that they are embroidered on a visible surface, but to fall far short of true ones—the motions, that are the real fastness and the real slowness [being expressed] by true numbers and all true figures, that are in relation to one another, and carry round the things in them; the motions are apprehended by reason and thought but not by sight.
>
> ταῦτα μὲν τὰ ἐν τῷ οὐρανῷ ποικίλματα, ἐπείπερ ἐν ὁρατῷ πεποίκιλται, κάλλιστα μὲν ἡγεῖσθαι καὶ ἀκριβέστατα τῶν τοιούτων ἔχειν, τῶν δὲ ἀληθινῶν πολὺ ἐνδεῖν, ἃς τὸ ὂν τάχος καὶ ἡ οὖσα βραδυτὴς ἐν τῷ ἀληθινῷ ἀριθμῷ καὶ πᾶσι τοῖς ἀληθέσι σχήμασι φοράς τε πρὸς ἄλληλα φέρεται καὶ τὰ ἐνόντα φέρει, ἃ δὴ λόγῳ μὲν καὶ διανοίᾳ ληπτά, ὄψει δ'ου.[67]

Although the passage is hard to translate and lends itself to a range of interpretations,[68] it is clear that Plato distinguishes the true motions (τῶν ἀληθινῶν)[69] from the actual motions of heavenly bodies. And he suggests that only the former are entitled to be objects of astronomy as a branch of mathematics, since they trace out true geometrical figures, and their inter-relations aptly expressible in terms of true numbers, that is, the

[67] *Ibid*., 529c7-d5.

[68] For the translation issue of this passage, see Adam (1929) pp. 186-187.

[69] Most commentators agree on translating 'τῶν ἀληθινῶν' as 'true motions', but it is not clear what 'true motion' might amount to, and the term has lent itself to a variety of interpretations. For various interpretations of 'true motions', see Seung (1996) pp.117-119. I accept the view that Plato's real astronomy is just pure mathematics; for Plato, the objects of astronomy are identical to the objects of pure mathematics such as pure geometry and arithmetic. This view is advocated by Seung and Mueller. See Seung (1996) pp. 118-119 and Mueller (1980).

ratio of the speeds of motions may be calculated in terms of true numbers;[70] in addition, these relations are not directly observable but may only be grasped only by reason.

Plato accepts that astronomers investigate the visible motions of stars. But he distinguishes such empirical astronomy from the real astronomy whose objects are the real motions mentioned above. Since empirical astronomy deals with sensible stars whose motions vary or are erratic, it does not qualify as a science.

If real astronomy does not treat the visible heavenly bodies, what is the role of their observation in astronomy? It would be difficult to deny that astronomy is based on such observation, or, at least, that observation is necessary for astronomy. In Plato's time, astronomy was a fairly descriptive science which made rapid advances on the back of accurate observations. For example, Eudoxus' *Phainomena* was no more than a description of the motions of visible stars.[71] Plato does not deny the role of observation

[70] What are the true numbers? Plato makes a distinction between pure numbers and material numbers: while a pure number is composed of absolutely equal units, an empirical number whose units are unequal, is a collection of sensible things. For this distinction between two different kinds of numbers, see *Philebus*, 56 d-e, 57c-e; *Republic,* VII, 525 c-e. It is notable that Aristotle also preserves this distinction (*Met.*, X, 1, 1052b35;1053a2; cf. *Phy*., IV, 11, 219b5-10).

For Plato, depending on which number it deals with, arithmetic is also divided into two kinds: arithmetic of the Many and philosopher's arithmetic. As the names indicate, the arithmetic of the Many is an empirical study of the phenomenal world which calculates the numbers of sensible things, while philosopher's arithmetic is a pure arithmetic which studies the characteristics of pure numbers not found in the empirical world. Since the study of pure arithmetic is independent of our experience, we may also call it *a priori* arithmetic.

Plato applies this dichotomy to other mathematical sciences such as geometry, astronomy, harmony, and optics, as well. See *Philebus*, 55d-57a, 57c; *Republic*, VII, 527a-b; 529d-e; 530a-b; 531b-c.

[71] See Lasserre (1964) pp. 144-168. Cf., *Met.*, XIII, 1073b3-17. The *Phainomena* is the earliest Greek work that describes star groups as constellations. The original is now lost.

in astronomy, but minimizes it by assimilating the motions of stars to illustrations in geometry:

> "Accordingly," said I, "we must use the embroidery of the heaven for our study of those [realities], just as if someone [would do who] chanced upon diagrams exquisitely drawn and worked out by Daedalus or some consummate artist. For anyone acquainted with geometry in some degree, seeing such things, would think that they are most beautifully finished by workmanship, but he would think it absurd to study them in all earnest for finding in such things the truth of equal or double or any other ratio."…I said, "Hence, using problems just as if we do geometry, we participate in the study of astronomy, and will let be the things in the heaven, if we really will make the intelligent part of the soul useful from its being useless by participating in the study of astronomy.
>
> Οὐκοῦν, εἶπον, τῇ περὶ τὸν οὐρανὸν ποικιλίᾳ παραδείγμασι χρηστέον τῆς πρὸς ἐκεῖνα μαθήσεως ἕνεκα, ὁμοίως ὥσπερ ἂν εἴ τις ἐντύχοι ὑπὸ Δαιδάλου ἤ τινος ἄλλου δημιουργοῦ ἢ γραφέως διαφερόντως γεγραμμένοις καὶ ἐκπεπονημένοις διαγράμμασιν. ἡγήσαιτο γὰρ ἄν πού τις ἔμπειρος γεωμετρίας, ἰδὼν τὰ τοιαῦτα, κάλλιστα μὲν ἔχειν ἀπεργασίᾳ, γελοῖον μὴν ἐπισκοπεῖν αὐτὰ σπουδῇ ὡς τὴν ἀλήθειαν ἐν αὐτοῖς ληψόμενον ἴσων ἢ διπλασίων ἢ ἄλλης τινὸς συμμετρίας...Προβλήμασιν ἄρα, ἦν δ᾽ ἐγώ, χρώμενοι ὥσπερ γεωμετρίαν οὕτω καὶ ἀστρονομίαν μέτιμεν, τὰ δ᾽ ἐν τῷ οὐρανῷ ἐάσομεν, εἰ μέλλομεν ὄντως ἀστρονομίας μεταλαμβάνοντες χρήσιμον τὸ φύσει φρόνιμον ἐν τῇ ψυχῇ ἐξ ἀχρήστου ποιήσειν.[72]

Once the perfection of the stars' motions is denied, astronomy loses its force as a counter example against AFS. By the time of Plato, astronomers had discovered the apparent

[72] *Republic,* VII, 529d7-730c1.

irregularity of planetary motions, indeed devoting much of their attention to accounting for this irregularity. For instance, they found from observation that planets such as the Moon, the Sun, Mars, Jupiter and Saturn revolve much more slowly from west to east, in the direction of the circle of the zodiac, than they do from east to west; and that Venus in the course of its synodic revolution draws the line of a saw-tooth. Since they commonly conceived the universe to be a sphere or hemisphere and stars to move on the surface of the sphere, tracing out a circle, it had become the astronomers' central preoccupation in those days to account for such irregularities, in dealing with which they typically had recourse to theories of the interactions between the motions of individual stars.

It is hard to imagine that Aristotle presented his argument from astronomy in ignorance of such astronomical facts—he refers to the systems of Eudoxus in his works.[73] Nevertheless, Aristotle's argument supposes the perfection of the stars' motion, possibly in consequence of Eudoxus' success in reconciling traditional assumptions with observational facts.[74] Eudoxus believed, on the one hand, like other contemporaries, that the motions of heavenly bodies are regular, uniform and circular, and on the other hand, was unable to discount his observation. It was one of Eudoxus' great astronomical achievements to reconstruct the original motions of stars by accounting for the irregularities in terms of the interplay between the rotations of four distinct spheres. In explaining the irregularities of the so called 'wandering stars,' Eudoxus posits four

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[73] Aristotle was after all one of the two greatest pupils of Eudoxus. He mentions Eudoxus' system at *DC.*, 291a29-291b11; *Met.* 1073b18-31. cf., 1073b3-17.

[74] Aristotle had metaphysical reasons for thinking that the motions of the heavenly bodies must be reducible to compositions of circular motion, since circular motion is the only sort capable of being eternal. See *DC.*, I, 2, 269a9-10; II, 3, 286a9-11; II, 4, 286b11-287a2.

distinct spheres, each of which has its own individual rotation. For example, he converts the saw-toothed curve which Venus traces out into the two twists of the hippopede[75] and again decomposes this hippopede into circular spherical motions meeting criteria of geometrical absoluteness.[76]

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[75] A *hippopede* is a figure of eight curve used by Eudoxus in his planetary theory. Hippopede means literally "foot of a horse." It is often known as the *hippopede of Proclus*, after Proclus who was the first to study it, together with Eudoxus, and also as the *horse fetter*.

"Eudoxus developed one of the first explanations of planetary movement which accounts for the observable phenomena of retrograde motion. The mathematical construct he developed uses two counter-rotating concentric spheres with offset axes of rotation. The combined motion of these spheres causes a planetary motion called a "hippopede", which when combined with the planet's normal easterly track across the backdrop of the fixed stars, causes the planet to move briefly in the opposite direction (URL=http://hsci.cas.ou.edu/images/ applets/ hippopede.html)."

[76] Eudoxus was probably the first astronomer who provided a mathematical explanation for the irregularities of the motions of the heavenly bodies. His system has two main distinguishable features: (i) he supposes not one, but four, different spheres for each planet and (ii) while he, like other astronomers, locates fixed stars on the surfaces of the spheres, he locates wandering stars between the surfaces of the spheres and the Earth. The following passage from Aristotle allows us to figure out how Eudoxus explains the motions of the stars in his system. "This discussion shows that their movements are faster or slower according to their distances. since it is admitted that the outermost revolution of the heaven is simple and fastest, whereas that of the other [inner spheres] is slower and composite (for each by its own revolution is going against [the motion of] the heaven), it is, then, reasonable that the star nearest to the simple and primary revolution completes its own revolution in the longest time and the one farthest away in the shortest, and with the others, the nearer one always in a longer time, and the farther one in a shorter time. For the nearest of all is most influenced [by the primary motion], and the farthest least, owing to its distance. Again, those between these two [are influenced] in proportion to their distances, as the mathematicians show (Συμβαίνει δὲ κατὰ λόγον γίγνεσθαι τὰς ἑκάστου κινήσεις τοῖς ἀποστήμασι τῷ τὰς μὲν εἶναι θάττους τὰς δὲ βραδυτέρας: ἐπεὶ γὰρ ὑπόκειται τὴν μὲν ἐσχάτην τοῦ οὐρανοῦ περιφορὰν ἁπλῆν τ᾽ εἶναι καὶ ταχίστην, τὰς δὲ τῶν ἄλλων βραδυτέρας τε καὶ πλείους (ἕκαστον γὰρ ἀντιφέρεται τῷ οὐρανῷ κατὰ τὸν αὑτοῦ κύκλον), εὔλογον ἤδη τὸ μὲν ἐγγυτάτω τῆς ἁπλῆς καὶ πρώτης περιφορᾶς ἐν πλείστῳ χρόνῳ διιέναι τὸν αὑτοῦ κύκλον, τὸ δὲ πορρωτάτω ἐν ἐλαχίστῳ, τῶν δ᾽ ἄλλων τὸ ἐγγύτερον ἀεὶ ἐν πλείονι,

It is not certain whether Plato knew of Eudoxus' new astronomical theory, since there is no passage in the *Dialogues* which refers unequivocally either to it or a theory modeled on it. If Plato was not then abreast of Eudoxus, his astronomical views may be ignored as being obsolete. But it is more likely that Plato was familiar with the theory and rejected it due to his antipathy to his contemporary astronomers who based their study on the observation of visible stars. More fundamentally, as we have seen before, Plato is deeply skeptical toward the possibility of a science of sensible things.

After all, if Aristotle's view on astronomy is accepted, astronomy would present a cogent counterexample to AFS. We have seen that Plato distinguishes pure astronomy from empirical astronomy. But the ground of the distinction is the imperfection of the heavenly bodies' motions. Now, since it is assumed that the motions of the stars trace out perfect geometrical figures, Plato's distinction between pure and empirical astronomy lacks any basis.

However, there is another problem. The first argument of AFS posited that, while the objects of sciences are simple, sensible particulars are complex.[77] For instance, geometry does not study triangular things such as a bronze isosceles triangle but the triangle itself; and while the triangle itself has only the necessary properties of triangle, a

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τὸ δὲ πορρώτερον ἐν ἐλάττονι. Τὸ μὲν γαρ ἐγγυτάτω μάλιστα κρατεῖται, τὸ δὲ πορρωτάτω πάντων ἥκιστα διὰ τὴν ἀπόστασιν: τὰ δὲ μεταξὺ κατὰ λόγον ἤδη τῆς ἀποστάσεως, ὥσπερ καὶ δεικνύουσιν οἱ μαθηματικοί) (*DC*., 291a32-291b10)."

For Arisotle's explanation of Eudoxus' four spheres, see *Met.,* XII, 8, 1073b18-31. Cf. *Ibid.*, 1073b3-17. See also Dicks (1970) pp.151-189. For Greek philosophical works treating ancient astronomy, see Heath (1932).

[77] *Phaedo,* 74ff. and 78ff.; *Republic,* V, 476e-479 and VII, 523a.

sensible particular triangular thing has other non-geometrical properties as well which are irrelevant to any geometrical study. Similarly, the visible motion of a star has not only mathematical properties, such as speed or the circular orbit dealt with by arithmetic or geometry, but also has other non-mathematical properties. As we will see later on, this is the problem which Aristotle purposes to solve by introducing abstraction. I will deal with this problem of complexity of particulars again when discussing his notion of abstraction.[78]

Although controversies remain over the finer points of Aristotle's criticism of Plato's philosophy of mathematics, overall his criticism of mathematical Platonism is persuasive and effective. Aristotle is the first philosopher who directly develops a criticism of mathematical Platonism. He invented a number of anti-Platonic arguments in the philosophy of mathematics, some of which (as below) survive into contemporary debates:

> The properties of numbers apply in a musical scale, in the heaven, and in many other cases. Those who assert that only mathematical number exists cannot say anything such as this according to their hypotheses; but they used to say that sciences will not be concerned with these [sensible] things. But we say that they are, as we said before. And it is obvious that mathematical objects are not separate; if they were separate, their properties would not apply to bodies.
>
> τὰ πάθη τὰ τῶν ἀριθμῶν ἐν ἁρμονίᾳ ὑπάρχει καὶ ἐν τῷ οὐρανῷ καὶ ἐν πολλοῖς ἄλλοις. τοῖς δὲ τὸν μαθηματικὸν μόνον λέγουσιν εἶναι ἀριθμὸν οὐθὲν τοιοῦτον ἐνδέχεται λέγειν κατὰ τὰς ὑποθέσεις, ἀλλ᾽ ὅτι οὐκ ἔσονται αὐτῶν

[78] See Chapter Two, §1.

> αἱ ἐπιστῆμαι ἐλέγετο. ἡμεῖς δέ φαμεν εἶναι, καθάπερ εἴπομεν πρότερον. καὶ δῆλον ὅτι οὐ κεχώρισται τὰ μαθηματικά: οὐ γὰρ ἂν κεχωρισμένων τὰ πάθη ὑπῆρχεν ἐν τοῖς σώμασιν.[79]

Thus, for Aristotle, Platonism cannot account for the applicability of mathematics to the sensible world. One way to explain the applicability of mathematical theory to the physical world would be to show that mathematics is a theory of a certain *aspect* of physical objects in the same way that physics is. But, since Platonists assert that mathematical objects are distinct from physical objects, they find themselves having to explain the relevance of mathematical objects to the physical world,[80] for instance, as teased out by Balaguer in terms of causal relations:

> For since Platonists maintain that mathematical objects exist outside of spacetime, they endorse what we might call the principle of causal isolation (PCI), which says that there are no causal interactions between mathematical and physical objects. But this gives rise to the following question: If there are no mathematical facts that are causally relevant to any physical facts, why is mathematical theory (which presumably is concerned with mathematical facts) relevant to physical theory (which presumably is concerned with physical facts)?[81]

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[79] *Met.*, XIV, 3, 1090a24-30. A similar passage can be found at *Ibid.,* 1090a9-15.

[80] "It is evident, then, both that the contrary theory will say the contrary, and that the problem raised just now, why is it that while [numbers] are in no way present in sensible things their properties are present in sensible things, must be solved by those who hold this view (δῆλον οὖν ὅτι καὶ ὁ ἐναντιούμενος λόγος τἀναντία ἐρεῖ, καὶ ὃ ἄρτι ἠπορήθη λυτέον τοῖς οὕτω λέγουσι, διὰ τί οὐδαμῶς ἐν τοῖς αἰσθητοῖς ὑπαρχόντων τὰ πάθη ὑπάρχει αὐτῶν ἐν τοῖς αἰσθητοῖς (*Met.*, XIV, 3, 1090b1-5))."

[81] Balaguer (1988) p. 110.

Platonists might argue for some kind of a relationship between Forms and physical objects: a physical object *participates in* or *imitates* a Form; and this relationship is downward only, that is to say, a Form, *F*-ness, is the cause of a particular thing's being *F*, but *F*-ness is causally independent of an *F*-thing; namely, it exists by itself. But as Plato himself recognizes, without further explanation of what such terms as 'participate' or 'imitate' means, the attempt to explain the relationship between a Form and its corresponding particular can be no more than *ad hoc* or a mere metaphor.[82]

[82] Plato's criticism of the early theory of Forms, see *Parmenides*, 129a-135b.

Chapter Two

Aristotle's Theory of Abstraction

1. The Need for Introducing Abstraction

Let us suppose that Aristotle's criticism of mathematical Platonism is successful. Aristotle may then have proved the validity of his anti-Platonism in mathematics, but would still be far from having secured the whole of his theory. The more important question for Aristotle's theory of mathematics is whether he can provide an account of mathematical truth which is coherent with his realistic view of the sciences and theory of truth.

Obviously, how mathematics can be true for Aristotle depends on what kind of theory of truth Aristotle is taken to have. On this score, Aristotle is faithful to 'what is (τὸ ὄν)' : "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, or of what is not that it is not, is true (τὸ μὲν γὰρ λέγειν τὸ ὂν μὴ εἶναι ἢ τὸ μὴ ὂν εἶναι ψεῦδος, τὸ δὲ τὸ ὂν εἶναι καὶ τὸ μὴ ὂν μὴ εἶναι ἀληθές)."[83] As such, Aristotle's problem concerning the philosophy of mathematics comes down to showing that mathematicals are some kind of existents in his ontology.

Since there are not only particular individuals but also universals in Aristotle's ontological inventory, he is in a better position to maintain scientific realism than

[83] *Met.*, IV, 7, 1011b26-27.

nominalists. However, the complexity of particulars remains a problem for him; while a science deals with *F*-itself, there is no *F*-itself separated from *F*-things in the Aristotelian world; and an *F*-thing is different from *F*-itself, because it has other properties than *F*-ness (and necessary properties of *F*-ness). Thus Aristotle needs to show that the *F*-itself which a science is talking about is nothing but a universal, *F*-ness, in particular *F*-things. He attempts to show this by proving that, although a universal, *F*-ness, cannot be ontologically separated from a particular *F*-thing, a legitimate conceptual separation does obtain between the universal and the particular by abstraction.[84]

In the remainder of this chapter, I first offer my own account of Aristotle's abstraction, and show how this abstraction solves the complexity problem of particulars. Next, I compare Aristotle's theory of abstraction with that of other empiricists in an attempt to reveal its distinctive characteristics. Finally, I consider the traditional interpretation of Aristotle's abstraction, examining texts that seemingly support the traditional view. The aim here is to show that the Aristotelian texts in questions do not decisively support the traditional view, and that my interpretation can still be maintained.

2. The origin of Aristotle's Abstraction

[84] Describing Aristotle's abstraction here and hereafter, I use the term 'concept' in the Fregean sense. For Frege, concepts are objective in the sense that they are mind-independent, and so are distinguished from anything subjective, any mental item or any property of a mental item. See Frege (1968) §§46-47.

The term 'abstraction', etymologically speaking, is the translation of the Greek word, 'ἀφαίρεισις' from the verb, 'ἀφαιρεῖν'. 'ἀφαιρεῖν' means literally 'to take from' or 'take away from.'[85] 'πρόσθεσις' is often used along with 'ἀφαίρεισις' to express the two contrary actions of adding and subtracting. For instance:

> The things which simply come to be come to be absolutely: some of them by change of shape, like a statue; some by addition (προσθέσει), like things which grow; some by subtraction (ἀφαιρέσει) as a Hermes [comes to be] out of the stone.
>
> γίγνεται δὲ τὰ γιγνόμενα ἁπλῶς τὰ μὲν μετασχηματίσει, οἷον ἀνδριάς, τὰ δὲ προσθέσει, οἷον τὰ αὐξανόμενα, τὰ δ' ἀφαιρέσει, οἷον ἐκ τοῦ λίθου ὁ ῾Ερμῆς.[86]

In Plato, 'ἀφαιρεω' and 'προστίθημι' refer to a kind of dialectical method, and may respectively be translated as 'subtracting' and 'adding':

> I often repeat it on purpose, in order that nothing may escape us, and that you may add (προσθῇς) or subtract (ἀφέλῃς) something if you wish. And Cebes said: "I do not want to add (πρσθεῖναι) or subtract (ἀφελεῖν) anything at present. And that is what I say."
>
> καὶ ἐξεπίτηδες πολλάκις ἀναλαμβάνω, ἵνα μή τι διαφύγῃ ἡμᾶς, εἴ τέ τι βούλει, προσθῇς ἢ ἀφέλῃς.΄ Καὶ ὁ Κέβης, ᾿Αλλ᾿

[85] For the etymology of 'ἀφαιρεῖν' see Mueller (1970) p160, and Cleary (1985) p. 18.
[86] *Phys.*, I, 7, 190b5-7.

> οὐδὲν ἔγωγε ἐν τῷ παρόντι, ἔφη, οὔτε ἀφελεῖν οὔτε προσθεῖναι δέομαι: ἔστι δὲ ταῦτα ἃ λέγω. [87]

It is in *Topics* where 'ἀφαίρεσις' is first used as a technical term. Again, 'ἀφαίρεισις' appears as one of a pair with 'πρόσθεσις'; and both feature as part of a dialectical method used to decide which of two items is the more desirable.[88] For example, when there are three different items, *a, b,* and *c*, and *a* and *b* are added to *c* separately, if the whole resulting from adding *a* to *c* is better than the whole of *b* and *c*, then *a* is more preferable.[89] Similarly, the rule of subtraction is also employed as a decision procedure. When two different objects of choice, *a* and *b* form part of *c* and each of *a* and *b* is subtracted from *c*, the element whose absence renders the remainder worse may be deemed the more desirable.[90] It is important that the origin of the terminology is traced to a dialectical context, since this provides a clue to the special characteristics of Aristotle's abstraction, which connotes a logical or linguistic method, rather than a private mental operation.

The fact that the word for 'abstraction' often appears as a pair with 'addition (ἀφαίρεσις)' throughout Aristotle's works is another indication of the operation's

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[87] *Phaedo,* 95e2-6. See also *Euthydemus*, 296b; *Cratylus*, 393d and 432a; *Parmenides*, 131d and 158c.

[88] Cleary (1985) p. 19.

[89] *Topics*., III, 3, 118b10-20.

[90] Aristotle's employment of subtraction and addition without further explanation seems to indicate that the methods were already familiar to his readers or audiences. For this point, see Cleary (1985) p.19. See also *NE.*, I, 7, 1097b16-21.

logical character. In fact, in most passages in which 'ἀφαίρειοις' appears as a technical term, it can better be translated by 'subtraction'.[91] Throughout Aristotle's oeuvre, indeed, the philosophical sense of addition always indicates a linguistic operation as a dialectical method. For instance, Aristotle asks whether a formula with an addition may count as a definition,[92] and argues that, since anything which is snub also must be a snub nose, the definition of 'snub' should include the formula of a snub *nose*. Thus, one can define 'snub' from an addition, namely, the definition of snub is the formula of a snub nose. Although it is not clear what exactly Aristotle means by 'the formula [constructed] from an addition' (τὸ ἐκ προσθέσεως λόγος),[93] it is obvious that the addition involves combining two concepts, namely, it is a linguistic operation. If its correlate term, addition, refers to a linguistic operation, it is reasonable to expect that 'subtraction' will also be a matter of the same sort of linguistic or logical method.

3. Abstraction—a Way to Find the Primitive Subject

I claimed earlier that Aristotle introduces abstraction to refute Plato's arguments from the sciences and to defeat mathematical Platonism. Abstraction as a method of decision,

[91] This is also true in the case of mathematical abstraction. Clearly chooses 'subtraction' as his official translation of 'ἀφαίρεσις.' This translation has an advantage of preventing other abstractionists' concept of abstraction confounded with Aristotle's ἀφαίρεσις. See Clearly (1985).

[92] *Met.*, VII, 5, 1030b14-1031a1. cf. *Apo.,* I, 27, 87a36; *Met.*, XIII, 4, 1079b4-10.

[93] See Bostock (1994), pp. 96-97. Another place where Aristotle mentions 'the formula from an addition' is *Met.*, VII, 4, 1029b18-19.

however, has nothing to do with anti-Platonic argument. However, we can find another type of abstraction in *Posterior Analytics*. Aristotle there employs abstraction as a way to find the primary subject of a given predicate. This sort of abstraction, I will show, is identical with the mathematical abstraction which Aristotle exploits in *Met.*, XIII and XV, to criticize AFS by rebuting mathematical Platonism. First, I will explain how abstraction is used to identify the primitive subject of a predicate, before expounding how such abstraction can be used to defeat Platonic scientific realism.

In *Posterior Analytics*, I, 5, Aristotle poses a question as to how we can confirm that a certain acquired knowledge of an object is universal. He asks, 'how we can know that every triangle has angles equal to the sum of two right angles'?[94] Aristotle argues that we do not know this theorem universally or absolutely, even if we prove it separately of each kind of triangle, until we know that the property belongs to them *qua* triangle, not *qua* isosceles, scalene or equilateral:

> …even if you prove for each [kind of] triangle, either by one or by another demonstration, that each has two right angles…you do not yet know of triangles that they have two right angles, universally nor even of every triangle…For you do not know it of triangles *qua* triangle…So when do you not know universally, and when do you know absolutely?
>
> …οὐδ᾽ ἄν τις δείξῃ καθ᾽ ἕκαστον τὸ τρίγωνον ἀποδείξει ἢ μιᾷ ἢ ἑτέρᾳ ὅτι δύο ὀρθὰς ἔχει ἕκαστον…οὔπω οἶδε τὸ τρίγωνον ὅτι δύο ὀρθαῖς…οὐδὲ καθ᾽ ὅλου τριγώνου…οὐ γὰρ ᾗ τρίγωνον οἶδεν, οὐδὲ πᾶν τρίγωνον… Πότ᾽ οὖν οὐκ οἶδε καθόλου, καὶ πότ᾽ οἶδεν ἁπλῶς.[95]

[94] *Apo.*, I, 5, 74a25-37.
[95] *Ibid.*, 74a25-33.

Two questions come up: First, what exactly does the '*qua*' mean? Second, how can we decide whether a property belongs to each triangle *qua* triangle, *qua* equilateral or *qua* isosceles? In fact, answers to both questions can be found in the text. Let us examine Aristotle's solution for the first question first.

Aristotle suggests abstraction as a way to decide whether a certain property belongs to an object *qua* A or *qua* B. For instance, given a bronze isosceles triangle, suppose that we need to decide whether the property, having angles equal to two right angles, belongs to it *qua* triangle or *qua* isosceles or *qua* bronze. To decide this, Aristotle proposes, we only have to remove (ἀφαιρεῖν) each of the physical triangle's predicates successively, e.g., being bronze, being isosceles, and being triangular, and check whether the property still holds; for instance, even if we remove 'being bronze' from a given triangle of this description, it has still angles equal to two right angles because it is not only a bronze isosceles triangle, but isosceles triangles in general which have angles of 180 degrees. Likewise, even after 'being isosceles' is removed, it will have the property of 'having angles of 180 degrees'. But if we remove or abstract (ἀφαιρεῖν) 'being a triangle' from the list of its predicates, it will no longer have the property of being equal to right angles. This confirms, Aristotle argues, that the property belongs to it *qua* triangle.

> It is clear that, after [other things] have been abstracted, it belongs to the primitive subject, e.g. two right angles will belong to bronze isosceles triangles, even after both being bronze and being isosceles have been removed. But not when figure or limit have been. But [two right angles do] not [belong to these items] primitively. Then, what is the primitive subject? If when triangle is abstracted, [two right angles do not belong to it], then, it is in virtue of this that it belongs to the other items as well, and it is to this that the demonstration applies universally.
>
> δῆλον ὅτι ὅταν ἀφαιρουμένων ὑπάρχῃ πρώτῳ. οἷον τῷ ἰσοσκελεῖ χαλκῷ τριγώνῳ ὑπάρξουσι δύο ὀρθαί, ἀλλὰ καὶ τοῦ χαλκοῦν εἶναι ἀφαιρεθέντος καὶ τοῦ ἰσοσκελές. ἀλλ᾽ οὐ τοῦ σχήματος ἢ πέρατος. ἀλλ᾽ οὐ πρώτων. τίνος οὖν πρώτου; εἰ δὴ τριγώνου, κατὰ τοῦτο ὑπάρχει καὶ τοῖς ἄλλοις, καὶ τούτου καθόλου ἐστὶν ἡ ἀπόδειξις.[96]

Since for Aristotle the question whether A belongs to B *qua* C is the equivalent to the question whether A is a property of B and C is the primitive subject of A,[97] abstraction can be regarded as a way to determine what a property belongs to not coincidentally, but primitively; e.g. 'having angles equal to two right angles' may belong to something brazen or isosceles coincidentally, but will belong primitively (ὑπάρχει πρώτων) to something triangular. Thus, 'triangle', which remains after other (less general) predicates of the whole compound thing have been removed, stands as the primary subject of the property of 'having angles equal to right angles'.

The order of abstraction is important in this procedure. As Aristotle himself recognizes, for instance, the property of 'having the sum of its internal angles equal to

[96] *Ibid.*, 74a37-74b4.
[97] *Ibid*., 6, 74a35-36.

two right angles' would disappear not only when the aspect of triangularity but also when the aspect of figure is removed from the bronze isosceles triangle.[98] But since the property does not belong to it *qua* figure, figure cannot be the primitive subject of that property. While 'having angles equal to two right angles' will not belong to a bronze isosceles triangle when 'being a figure' is abstracted, that property does not belong to a bronze isosceles triangle in virtue of its being a figure; it is obvious that not every figure has angles equal to two right angles, but any triangle does. This means that the order of subtraction (abstraction) is crucial in deciding the subject to which properties belong primitively. Notice that in the illustration with the bronze isosceles triangle, Aristotle subtracts each predicate in order of increasing generality, i.e. first bronze, through to isosceles, triangle, and figure.[99]

So far we have taken it for granted that if we remove or abstract (ἀφαιρεῖν) 'being a triangle' from the list of predicates of a bronze isosceles triangle, the property of 'having angles equal to two right angles' is no longer guaranteed to pertain to it. This was the rationale of thinking that the property of 'having angles equal to two right angles' belongs to a bronze isosceles triangle *qua* triangle. But on what ground can we say that if something is not triangular, then it does not always have the property of having 180 degree angles? In this case, we know that there are counterexamples that not everything which is brazen and isosceles has angles summing to 180 degrees. But it is not always easy to find such counterexamples. Moreover, Aristotle does not suggest any way to find

[98] Clearly (1985) p. 23.
[99] For more discussion of the order of subtraction, see Cleary (1985) p.23; Barnes (1994) p. 125.

such counter examples; nor does he even mention finding a counterexample in relation to his abstraction. In order to determine the character of abstraction, i.e., whether it is a private mental process or a sort of conceptual analysis, it is important to know more about its way of deciding whether a property belongs to an object after the removal from it of certain predicates; abstraction is an integral part of such a decision process.

We can begin to deal with this theme by considering what Aristotle means in saying that A belongs to B primitively or that B is the primitive subject of A. In the example of the bronze isosceles triangle, 'having angles equal to two right angles', which primitively belongs to a triangle, will always belong to a triangle. Since Aristotle defines 'coincidentally F' as 'F but neither always nor usually F,'[100] we can infer that, if A belongs to B primitively, A belong to B not coincidentally, and *vice versa*. Coincidence is also elsewhere in Aristotle opposed to necessity.[101] A primitive property—let us call a property, F, a primitive property of an object, *a,* iff F primitively belongs to *a, or a* is the primitive subject of F—is opposed to a coincidental property in this second sense too. Aristotle certainly believes that the proposition that the property, 'having angles equal to two right angles,' belongs to triangles is demonstrable;[102] and what is understandable in

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[100] *Met.*, VI, 2, 1026b27-33. Here, the term, 'coincidental' is the translation of 'συμβεβηκός.' So 'κατὰ συμβεβηκός' means by coincidence or coincidentally and 'τὸ συμβεβηκός' means something coincidental or a coincidental property. See Kirwan (1993) pp. 76-77.

[101] *Met.*,XI, 8, 1065a1-2. For another meaning of 'συμβεβηκός,' see Chapter Two, §3, n. 109.

[102] *Apr.*, I, 35, 48a29-40.

virtue of demonstrative understanding is necessary.[103] Thus, from these two facts, it can be inferred that by 'A is the primitive subject of B' or 'B belongs to A primitively' Aristotle means 'A belongs to B not coincidentally; since he opposes 'coincidentally' to 'always' or 'necessarily', it could be also said that by 'A primitively belongs to B' he means 'A always or necessarily belongs to B'.

A problem of this interpretation is that it includes all A's necessary properties as its primitive properties. But even if we remove 'being a triangle' from a certain figure's predicates, some of those necessary properties will still belong to it. For instance, although 'being a figure' always and necessarily belongs to a triangle, it does not belong to a bronze isosceles triangle *qua* triangle. So we need to narrow down the range of the primitive properties, while preserving two opposite senses of 'coincidental', i.e., 'always' and 'necessary'.

We find, in this connection, that Aristotle frequently also uses 'by coincidence (κατὰ συμβεβηκός)' to mean something opposite to 'in its own right (καθ' αὑτό).' Aristotle uses the expression, 'τὰ καθ' αὑτὰ ὑπάρχοντα' or 'τὰ καθ' αὑτὰ παθήματα (properties in its own right or *kath' hauta* properties) 'as a technical term with a contrasting sense to 'τὰ συμβεβηκότα (coincidental property)'. While coincidental properties (τὰ συμβεβηκότα) are not necessary,[104] properties in their

[103] *Apo.*, I, 3, 72b19-25; 5, 74b5-12; *Phys*, II, 200a15-19. For the proof of this theorem, see *Element*, I, 32. But it is not clear how 'having angles equal to two right angles' might fit with Aristotle's account of properties in its own right.
[104] *Apo.*, I, 6, 74b12-13.

own right must be held necessary to their subjects;[105] while there is no science of the coincidental because "all science is of that which is always, every demonstrative science is concerned with properties in its own right."[106] It is noteworthy here that the primitive properties which are constrasted with coincidental properties have a far narrow range than necessary properities. Further, a science does not study all the necessary properties of its objects; for instance, it is obvious that geometry does not demonstrate all the necessary properties of a triangle, such as being existent,[107] being capable of being thought, being a figure, etc. This distinguishes *kath' hauta* properties as objects of sciences from necessary properties.[108] Moreover, the fact that *kath' hauta* properti*es* are the objects of scientific demonstration provides a ground on which the primitive properties may be identified as *kath' auto* properties. We have seen earlier that Aristotle used 'having angles equal to two right angles' as an example of a primitive property of a triangle as an illustration of a mathematical demonstration.[109] The inference that 'A

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[105] *Ibid.*, 4, 74a7-8.

[106] *Ibid.*, 74b5-12; 10, 76b13

[107] Of course, for Aristotle, being (εἶναι) is not a property. Cf. *Apo.,* II, 7.

[108] This is Ross' view. See Ross (1924), pp. xciv-xcvii.

[109] *Apr.*, I, 35, 48a29-40. An issue involved in the identification of primitive properties with *kath' hauta* properties is whether Aristotle's own example of a primitive property of a triangle, 'having angles equal to two right angles,' can be legitimately among the *kath' hauta* properties of a triangle. Aristotle calls the property, 'having angles equal to two right angles' as 'τὸ καθ' αὑτὸ συμβεβηκός' of a triangle. Since he takes that property as an example of a primitive property of a triangle, if primitive properties can be identified with *kath' hauta* properties, a τὸ καθ' αὑτὸ συμβεβηκός of a triangle should be a member of the group of *kath' hauta* properties of a triangle.

Nevertheless, there are a couple of problems in identifying τὸ καθ' αὑτὸ συμβεβηκός with a *kath' hauto* property. The first problem is that, as the term, belongs to *a* primitively' means 'A is a *kath' hauto* property of *a'* finds further confirmation in the following passage:

'τὸ καθ᾽ αὑτὸ συμβεβηκός', indicates, Aristotle might seem to sort it out as a kind of coincidental property. But, according to Aristotle, "τὸ συμβεβηκὸς (the coincidental) is what happens but not always nor of necessity (τὸ συμβεβηκὸς ἔστι ὃ γίγνεται μέν, οὐκ ἀεὶ δ᾽ οὐδ᾽ ἐξ ἀνάγκης (*Met.,* XI, 8, 1065a1-2))," whereas *kath' hauta* properties are necessary to their subject. However, for Aristotle, 'τὸ συμβεβηκός' is sometimes used to mean just a property in general. 'συμβεβηκός' is derived from 'συμβεβηκέναι', the perfect form of the verb, 'συμβαίνειν', which literally means 'come together' without any necessary suggestion that that this *coming together* must be *not by necessity*. Thus, 'τὰ καθ' αὑτὰ συμβεβηκότα' may be translated into 'accidental properties in its own right,' but may mean also just 'properties in its own right'. Adopting the second reading, 'τὰ καθ᾽ αὑτὰ συμβεβηκότα' can be seen as an expression referring to *kath' hauta* properties along with other expressions such as 'τὰ καθ᾽ αὑτὰ ὑπάρχοντα' and 'τὰ καθ᾽ αὑτὰ παθήματα'. There are passages in which obviously the latter reading is the more appropriate (especially at 75a18-19 and 75b1-2). Further, the characteristics of *kath' hauta properties* described by Aristotle at 74b5-12 bear comparison to those of τὸ καθ᾽ αὑτὸ συμβεβηκός: *kath' hauta* properties are what a demonstrative science or knowledge is concerned with and they are necessary to their subject. Likewise, τὰ καθ᾽ αὑτὰ συμβεβηκότα necessarily (or universally) belong to their subject and they are objects of scientific demonstrations. We have seen earlier that Aristotle used the property, 'having angles equal to two right angles', as an example of τὸ καθ᾽ αὑτὸ συμβεβηκός, by way of illustrating a mathematical demonstration (*Apr*., I, 35, 48a29-40); 'τὰ καθ᾽ αὑτὰ συμβεβηκότα' are said to be what demonstrations make plain (75b2-3). The second problem, though, is more serious. Although Aristotle gives us an account of two senses of belonging *kath' hauto*, 'having angles equal to two right angles', his own example of τὸ καθ᾽ αὑτὸ συμβεβηκός, does not seem to fit into either of these two senses: the account of what is 'having angles equal to two right angles' does not requires the concept 'triangle', nor does it appear in the definition of a triangle. Admitting that Aristotle's account leaves us uncertain about what Aristotle had in mind on the proper classification of 'having angles equal to two right angles, Tiles suggests that the second type of *kath' hauta* predications of a triangle enter as premises into the demonstration of the theorem that a triangle has angles equal to two right angles (see Tiles (1983) pp, 10-11). This arguably works despite the fact that 'having angles equal to two right angles' does not fit into either of Aristotle's two accounts of *kath' hauto.* Tiles' paper, "Why the Triangle has Two Right Angles *Kath' Hauto*" deals with this issue (see Tiles (1983)).

Let "of every case" and "in its own" be defined in this way. I call universal what holds of every case and in itself and *qua* itself. It is evident that, therefore, that whatever is universal belongs from necessity to its objects. To belong in its own and *qua* itself are the same thing, e.g., point and straight belong to line in itself for they belong to it *qua* line, and two right angles belong to triangle *qua* triangle for the triangle is in its own equal to two right angles.

Τὸ μὲν οὖν κατὰ παντὸς καὶ καθ᾽ αὑτὸ διωρίσθω τὸν τρόπον τοῦτον: καθόλου δὲ λέγω ὃ ἂν κατὰ παντός τε ὑπάρχῃ καὶ καθ᾽ αὑτὸ καὶ ᾗ αὐτό. φανερὸν ἄρα ὅτι ὅσα καθόλου, ἐξ ἀνάγκης ὑπάρχει τοῖς πράγμασιν. *τὸ καθ᾽ αὑτὸ δὲ καὶ ᾗ αὐτὸ ταὐτόν*, οἷον καθ᾽ αὑτὴν τῇ γραμμῇ ὑπάρχει στιγμὴ καὶ τὸ εὐθύ καὶ γὰρ ᾗ γραμμή, καὶ τῷ τριγώνῳ ᾗ τρίγωνον δύο ὀρθαί καὶ γὰρ καθ᾽ αὑτὸ τὸ τρίγωνον δύο ὀρθαῖς ἴσον.[110]

In this passage, Aristotle equates what belongs καθ᾽ αὑτό with what belongs to ᾗ αὐτό (τὸ καθ᾽ αὑτὸ δὲ καὶ ᾗ αὐτὸ ταὐτόν). The example clarifies what the statement that τὸ καθ᾽ αὑτὸ δὲ καὶ ᾗ αὐτὸ ταὐτόν means: 'having a point' and 'being straight' are καθ᾽ αὑτο properties of a line because they belong to line *qua* line. And it has been shown that A belongs to B *qua* B if and only if A is a primitive property of B.

What does it mean for A to be a *kath' hauto* property of B or that A belongs to B *kath' hauto*? The most illuminating passage regarding the nature of *kath' hauta*

[110] *Apo*., I, 4, 73b25-32.

properties is in *Posterior Analytics*, where Aristotle suggests two criteria for a *kath' hauto* property. Aristotle explains that A belongs to B *kath' hauto,*[111]

(1) if A appears in the definition of B, or

(2) if A is a property of B and B appears in the definition of A.

Aristotle illustrates the first sense by the example of lines and triangles: since the definition of triangle contains the concept of 'line',[112] 'line' belongs to 'triangle' in its own right. To explain his second sense, Aristotle takes two examples: (i) number and its properties such as odd, even, prime, composite, equilateral and oblong, and (ii) line and its properties such as being curved and being straight. It is impossible to define 'odd' or 'prime', for instance, without making some reference to 'number,' since they are properties of number, so that number appears as a definiens in the definition of those properties.[113]

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[111] Aristotle's account of the meaning of '*kath' hauto*' can be found at *Apo.*, I, 4, 73a34-73b17; 6, 74b7-10; 22, 83b13-18; *Met.,* V, 18, 1022a14-35. All these passages commonly explain these two senses of '*kath' hauto.*' At 73a34-38 and 1022a14-35, other meanings of '*kath' hauto'* are also found, which are irrelevant to our present discussion of the meaning of 'primitive property.' For instance, the distinction between καθ' αὑτο and συμβεβηκός in the third use of *kath' hauto* at 73a34-38 corresponds to the distinction between substance and the other categories of beings in *Categories;* its fourth use lends itself to a distinction of two different kinds of events. Even if, however, we admit those other usages of the term, the fact that the process of selecting *kath' hauta* properties has a fundamentally linguistic character will remain the same.For a full discussion of the four meanings of '*kath' hauto'* refer to Barnes (1994) pp, 112-117.

[112] See *The Elements*, I, Def. 19.

[113] Aristotle says that "odd belongs to number and number itself inheres in its account of number." (*Apo*., I, 22, 84a15-17). See also Barnes (1994) pp, 180-181.

Thus once we know the definition of A and B, we can have the list of *kath' hauta* properties of each A and B. How we can attain to a definition of each item is another matter. But after attaining to this definition, the process of eliciting *kath' hauta* properties from the definition is purely linguistic (or conceptual in the sense that the process is totally constituted by conceptual analysis). Since the process plays a central role in Aristotle's abstraction, his abstraction may also be characterized as being linguistic. Let us recap what has been argued about abstraction.

In *Posterior Analytics*, Aristotle introduces abstraction as an apparatus for determining the truth value of the following form of sentence:

(i) A belongs to B *qua* C.

It was shown that we can determine whether (i) is the case or not by means of abstraction. That is, A belongs to B *qua* C if and only if:

(ii) A does not always belong to B when C is abstracted or removed from the list of predicates of B.

This procedure, however, threw up the problem of how to decide if A always belongs to B or not, when C was abstracted. Since Aristotle regards (i) as being equivalent to:

(iii) A belongs to B, and A belongs to C primitively (or C is the primitive subject of A).

in order to know whether A does not always belong to B, when C is abstracted from the list of predicates of B, we only have to establish whether (iii) holds. C, then, will be established as the primitive subject of A if and only if

(iv) A is a *kath' hauto* property of C.

Since, given the definition of C, the procedure to decide whether (iv) true or not, is constituted of an analysis of relations between relevant concepts, we can likewise determine the truth of (ii) by the same form of conceptual analysis.

When Aristotle's abstraction is understood as a part of the process of selecting a certain group of properties, the '*qua*' operator, in the form of sentence, '*a* conceives of x *qua* y,' can be regarded as a kind of function, whose input is an ordered pair, <x, y> and its value is a set of *kath' hauta* proprites of y which belong to x. And the role of abstraction is to determine which property of x is a *kath' hauto* property of y. Thus, in the given sentence, '*a* conceives of x *qua* y,' abstraction has nothing to do with the verb, 'conceives of.' I will come back to this point when examining the traditional interpretation of Aristotle's abstraction to the effect that abstraction is a kind of private mental operation.[114]

4. Abstraction and Aristotle's Criticism of Mathematical Platonism

This interpretation of abstraction so far suggested is not only textually supported, but boasts the further advantage of making his philosophy avoid those difficulties that beset traditional abstractionist philosophies.[115] An important question, though, is whether Aristotle's theory of abstraction fulfils his purpose in introducing it.

I said earlier that Aristotle introduced abstraction as a logical apparatus to rebut Plato's AFS. The problems which Aristotle needs to solve regarding AFS are as follows: on the one hand, he needs to show that it is not necessary to posit separate scientific objects apart from sensible particulars; while, on the other hand, he also needs to show that the objects of sciences are objectively real, namely, they exist independently of our

[114] See Chapter Three, §6.
[115] "We attend less to a property and it disappears. By making one characteristic after another disappear, we get more and more abstract concepts...Inattention is a most efficacious logical faculty; presumably this accounts for the absentmindedness of professors. Suppose there are a black and a white cat sitting side by side before us. We stop attending to their color and they become colorless. We stop attending to their color and they become colorless, but they are still sitting side by side. We stop attending to their posture and they are no longer sitting (though they have not assumed another posture) but each one is still in its place. We stop attending to position; they cease to have place but still remain different. In this way, perhaps, we obtain from each one of them a general concept of Cat. By continued application of this procedure, we obtain from each object a more and more bloodless phantom. Finally we thus obtain from each object a something wholly deprived of content; but the something obtained from one object is different from the something obtained from another object—though it is not easy to say how (Frege (1952) pp. 84-85) ."

mind. Let us, first, consider how his theory of abstraction provides a solution for the first problem.

In AFS, Plato argued for the separate existence of objects of sciences on the basis of differences between objects of sciences and sensible particulars. There were two main differences between objects of these groups: First, while the objects of sciences maintain their identities cross-temporally, sensible particulars did not have such cross-temporal identities since they were in constant change over time. Since sensible particulars are indefinite in this sense, they elude definition. Aristotle avoids this problem of the indefiniteness of sensible particulars by making a distinction between essential and accidental properties: insofar as a sensible individual thing does not undergo its substantial changes, it maintains its identity. Even if its accidental properties change, we have still the same answer to the question of what the thing is. Thus, we can have a definition of a sensible object insofar as its essential properties are not changed. Nevertheless, this distinction between accidental properties and essences leaves another problem unresolved: A sensible object does not have self-identity because of its complexity. A triangle considered on the basis of geometrical theorems does not have such properties as being 'branzen' or 'isosceles' other than 'triangular,' which a particular bronze isosceles triangle has. Because of this complexity of sensible particulars, while we can have the definition of a triangle itself, there cannot be a definition of a particular bronze isosceles triangular thing; in the case of a sensible particular triangular thing, we may thus arrive at several different answers to the question of what it is.

It is this complexity problem for which Aristotle introduces abstraction as a solution. Aristotle argues that the complexity of a sensible particular does not imply the separate existence of objects of sciences. According to Aristotle, a geometer studies not the triangle itself separated from any sensible triangular thing, but a bronze isosceles triangle *qua* triangle. But what does he exactly mean when he says that a geometer studies a particular isosceles triangle *qua* triangle? We can consider the following formula:

(F1) A studies B *qua* C.

Suppose that 'A' represents a certain science, 'B' a particular sensible object, and C one of the objects of the science, respectively. Since to study an object is to study what properties the object has, it can be generally said that the objects of A are B's properties. But A does not study all the properties of B; for instance, coincidental properties are excluded in that a science does not concern itself with purely coincidental properties.[116] Nevertheless, it was also argued that not every necessary property belongs to B *qua* C.[117] We now, by means of abstraction, can determine which properties are to be selected, i.e.,

[116] Aristotle argues that there is no science of the coincidental "for all science is of that which is always or for the most part, but the coincidental is neither of these classes (ἐπιστήμη μὲν γὰρ πᾶσα τοῦ ἀεὶ ὄντος ἢ ὡς ἐπὶ τὸ πολύ, τὸ δὲ συμβεβηκὸς ἐν οὐδετέρῳ τούτων ἐστίν) (*Met.,* XI, 8, 1065a4-6)." Although τὰ καθ' αὑτὰ συμβεβηκότα are the properties with which a science will have a concern, I argued earlier that the phrase is likely to refer to just *kath' hauta* properties rather than some kind of coincidental properties.

[117] See Chapter Two, §3.

properties which do not always belong to B when C is abstracted or removed from B. Since a property belongs to B *qua* C if and only if it belongs to B and belongs to C primitively, (F1) can be rewritten as:

(F2) A studies such properties of B that belong primitively to C.

But we also know that a property, F, primarily belongs to C if and only if F is a *kath' hauto* property of C. So, the original formula amounts to saying that:

(F3) A studies *kath' hauta* properties of C which belong to B.

For example, if physics studies Socrates *qua* moving thing, it studies those properties of Socrates which are the *kath' hauta* properties of a moving thing. So, as Lear put it, '*qua*' operator filters out all the other predicates except the predicates which refer to the *kath' hauta* properties of C, the term which follows the *qua* operator.[118]

Plato argues that an object of a science is ontologically separated from sensible particulars on the basis of particulars' complexity. According to Plato, an object of a science, say C, has only properties necessary to itself, e.g., a triangle studied in geometry does not have such properties as color, weight, etc. which are coincidental to a triangle. Whereas, any sensible particular, B, will also have other properties which are not necessary to C, e.g., a bronze sphere has non-geometrical properties as well. While C can

[118] Lear (1982) pp. 168-175.

be identified with only C itself and nothing else, B can be identified with something other than C, say, D, insofar as B has the essential property of D as well. Since B cannot be equal to C, Plato maintains, C must exist apart from B if C exists at all. But, according to Aristotle's analysis, when A studies B *qua* C, C is only one of many true predicates of B, and the objects of A, namely, *kath' hauta* properties of C, are nothing but certain properties of B. Since C only has to be a predicate of A, B does not have to be identical with C; so that the complexity of B, which was the ground of the non-identity between B and C, cannot be a reason to posit the separate existence of C from B.

The fact that a scientific study, A, takes account exclusively of the *kath' hauta* properties of C among the properties of B makes it appear as though A deals with a separate entity, namely, C. But as we have seen, such separation is conceptually possible by means of abstraction. Since such abstraction is conceptual, in Aristotle's view, the separation is not real, but only conceptual; it is only for the sake of investigation. He says:

> Each [question] would be best investigated in this way—by supposing what is not separate as being separate, as the arithmetician and the geometer do. For a man *qua* man is one and indivisible; and [the arithmetician] supposes it as one and indivisible, and then considers whether something belongs to a man *qua* indivisible. But the geometer [supposes him] neither *qua* man nor *qua* indivisible, but *qua* a solid. For even if he had not been indivisible, it is evident that some [properties] would have belonged to him apart from being such things…
>
> ἄριστα δ' ἂν οὕτω θεωρηθείη ἕκαστον, εἴ τις τὸ μὴ κεχωρισμένον θείη χωρίσας, ὅπερ ὁ ἀριθμητικὸς ποιεῖ καὶ ὁ γεωμέτρης. ἓν μὲν γὰρ καὶ ἀδιαίρετον ὁ ἄνθρωπος ᾗ

> ἄνθρωπος: ὁ δ᾽ ἔθετο ἓν ἀδιαίρετον, εἶτ᾽ ἐθεώρησεν εἴ τι τῷ ἀνθρώπῳ συμβέβηκεν ᾗ ἀδιαίρετος. ὁ δὲ γεωμέτρης οὔθ᾽ ᾗ ἄνθρωπος οὔθ᾽ ᾗ ἀδιαίρετος ἀλλ᾽ ᾗ στερεόν. ἃ γαρ κἂν εἰ μή που ἦν ἀδιαίρετος ὑπῆρχεν αὐτῷ, δῆλον ὅτι καὶ ἄνευ τούτων ἐνδέχεται αὐτῷ ὑπάρχειν...[119]

Since it turns out that the objects of A are certain properties of B, there is no need to posit C's separate existence. Thus, Aristotle can solve the problem of complexity by way of abstraction. In addition, he can also maintain his scientific realism; as beings in categories other than substance, all the properties of A are real properties for Aristotle.

5. Abstraction as a Way of Obtaining Universals

Throughout the history of philosophy, abstraction has been regarded by empiricists as a nominalistic way of explaining how we acquire general concepts. For example, in his *Essay*, Locke says that particular ideas are made to represent all objects of the same kind by the mind's separating such ideas from other existence and from the circumstances of real Existence.[120] In a Lockean sense general ideas (universals) are the product of abstraction;[121] and abstraction is *a mental operation* by which what is common among

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[119] *Met*., XIII, 3, 1078a21-28. See also *Phys*., II, 1, 193b23.

[120] Locke (1975) BK. II, Ch. 11, §9. See also Yolton (1993), p. 7.

[121] A tension exists, though, between his ontology of particulars and the existence of general ideas. Berkeley later objects that if all things that exist are particular, *general* ideas cannot exist. But, in other places, Locke argues that ideas are also particulars, namely, there is no general idea;

particular ideas is retained and what make each idea differ, such as time and place, are left out.[122]

Berkeley uses the term ‘abstraction’ in a similar way, further dividing, however, into two kinds. The first kind of abstraction concerns the mind’s conceiving each quality singly apart from those other qualities with which it is united. [123] The second involves the obtaining of a general idea (such as ‘man’ or ‘book’) from particulars through the mind’s leaving out of those particularities that distinguish them one from another, retaining only what is common to all. The second abstraction corresponds exactly to Locke’s; and the basic meaning of abstraction is preserved: something common is retained and something else is left out.[124] The second form of abstraction would appear essential in the constitution of general terms such as ‘book’, ‘animal’, and ‘man’. But, unlike Locke, Berkeley denies that there are such general ideas acquired through abstraction. It is notable, however, that even the first type of abstraction still refers to a subjective mental process; it entails a deliberate lack of attention to certain circumstantial features of objects. Since Hume is committed to the same particularist ontology as Berkeley, he arrives at a similar view of abstraction and the existence of general ideas.

Whether or not these thinkers agree on the existence of general ideas, they all understand abstraction as (i) a subjective mental operation, and (ii) an epistemic process

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generality is a certain way of the mind’s conceiving things or ideas (Locke (1975) BK. III, Ch. 3, §6 and§11).

[122] *Ibid.*, BK. II, Ch. 11, §9.

[123] Berkeley (1710) Intro. §7.

[124] But, unlike Locke, Berkely explicitly denies the existence of general or abstract ideas, which general terms stand for. See *Ibid*., Intro. §10.

by which we obtain universals from particulars. This meaning of abstraction was generally retained up to Frege and may here be dubbed epistemic abstraction.[125] And such a form of abstraction has usually been a way for nominalists to provide a semantics for universal concepts without committing themselves to the existence of Platonic entities. The usual nominalist strategy has been to seek to reduce universals in ontological terms to the products of the mental activity of abstraction.

The interpretation of Aristotle's abstraction has been significantly affected, even warped, by the prevalent understanding of abstraction in modern philosophy. Traditionally, commentators have interpreted Aristotle's abstraction as the process through which the mind acquires a universal from particulars. For instance, Thomas comments that universals are abstracted from particulars by abstraction in Aristotle.[126] At the same time, though, more recent commentators have begun to query this traditional interpretation. These commentators tend to interpret Aristotle's abstraction as a linguistic or logical process, rather than a psychological;[127] presumably, this tendency is due to Frege's devastating criticism of psychologism, a kind of subjectivism in the philosophy of mathematics. Overall, I agree that the recent commentators' line of interpretation is the correct way to understand Aristotle's abstraction, although I would differ from them in some of my emphases and readings of particular passages and arguments.

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[125] Frege (1968) §§ 45 and 61.

[126] "…in one way as we understand universals abstracted from singulars, and in another way as we understand the objects of mathematics abstracted from sensible things… (Aquinas (1961) §158, p. 65).

[127] See Cleary (1985); Mueller (1970) pp. 159-163; Lear (1982) pp. 168-175.

To work out the detail of this reading, though, we would do well, first of all, to revisit some of the passages in Aristotle that would seem to lend support to the traditional interpretation. We should also pause before the fact that Aristotle would seem to introduce abstraction with a similar motivation to that of the nominalists, i.e., to provide a semantics for the universal concepts used without positing Platonic entities; this has led to an assimilation of his abstractions to that of the nominalists. Unfortunately, those who interpret Aristotle's abstraction as a linguistic analysis have not provided us with either any analysis of those passages or any argumentation against the traditional view.

The first passage we will look at is the one which some commentators have taken to imply that universals are the product of abstraction. [128] This is from *Posterior Analytics*:

> It is evident also that if any perception has been lost, some knowledge must be also lost, which it is impossible to get since we learn either by induction or by demonstration. And demonstration proceeds from universals and induction from particulars; but it is impossible to consider universals except through induction since one will be able to make familiar even *the things said as a result of abstraction* through induction that some things belongs to each genus, even if they are not separate, in so far as each thing is [considered] *qua* such, and it is impossible to get an induction without having perception, for perception is of particulars; it is impossible to gain scientific knowledge of them; for [it can be gained] neither from universals without induction, nor through induction without perception.
>
> Φανερὸν δὲ καὶ ὅτι, εἴ τις αἴσθησις ἐκλέλοιπεν, ἀνάγκη καὶ ἐπιστήμην τινὰ ἐκλελοιπέναι, ἣν ἀδύνατον λαβεῖν,

[128] See Mueller (1970) p. 160.

> εἴπερ μανθάνομεν ἢ ἐπαγωγῇ ἢ ἀποδείξει, ἔστι δ᾽ ἡ μὲν ἀπόδειξις ἐκ τῶν καθόλου, ἡ δ᾽ ἐπαγωγὴ ἐκ τῶν κατὰ μέρος, ἀδύνατον δὲ τὰ καθόλου θεωρῆσαι μὴ δι᾽ ἐπαγωγῆς (ἐπεὶ καὶ *τὰ ἐξ ἀφαιρέσεως λεγόμενα* ἔσται δι᾽ ἐπαγωγῆς γνώριμα ποιεῖν, ὅτι ὑπάρχει ἑκάστῳ γένει ἔνια, καὶ εἰ μὴ χωριστά ἐστιν, ᾗ τοιονδὶ ἕκαστον), ἐπαχθῆναι δὲ μὴ ἔχοντας αἴσθησιν ἀδύνατον. τῶν γὰρ καθ᾽ ἕκαστον ἡ αἴσθησις: οὐ γὰρ ἐνδέχεται λαβεῖν αὐτῶν τὴν ἐπιστήμην: οὔτε γὰρ ἐκ τῶν καθόλου ἄνευ ἐπαγωγῆς, οὔτε δι᾽ ἐπαγωγῆς ἄνευ τῆς αἰσθήσεως.[129]

In this passage, Aristotle seems to assume that the 'things said as a result of abstraction (τὰ ἐξ ἀφαιρέσεως λεγόμενα)' are universals; because what is abstracted is grasped by induction, and what is grasped by induction is a universal. Thus, it is plausible to suppose that 'the things said as a result of abstraction' will be universals insofar as they are grasped by induction. But does the passage mean to say that universals are acquired by abstraction? From the fact that some things abstracted are universals, it does not follow that universals are acquired by abstraction;[130] in the passage, it is not abstraction, but induction by which a universal is acquired from particulars. In other words, it is quite possible that universals have been already acquired by induction before they are abstracted. In another passage, Aristotle again adduces induction as the epistemological process by which a general concept is obtained from particulars:

[129] *Apo.*, I, 17, 81a38-18; 81b9.

[130] There is an ambiguity involved here, in that it is not clear whether by universals Aristotle means universal propositions or universal concepts. But it is more likely that 'universals' in context means universal concepts or universal properties, because, for Aristotle, a proposition is not an object of abstraction.

When one of the undifferentiated things makes a stand, there is a primitive universal in the soul; for although you perceive particulars, perception is of universals, e.g., of man, not of Callias the man. Then, a stand is made among such things, until something partless and universal makes a stand, e.g., such an animal, until animal does; and with animal a stand is made in the same way. Thus, it is evident that it is necessary for us to acquire knowledge of the primitives by induction; for perception produces universals in this way.

στάντος γὰρ τῶν ἀδιαφόρων ἑνός, πρῶτον μὲν ἐν τῇ ψυχῇ καθόλου (καὶ γὰρ αἰσθάνεται μὲν τὸ καθ᾽ ἕκαστον, ἡ δ᾽ αἴσθησις τοῦ καθόλου ἐστίν, οἷον ἀνθρώπου, ἀλλ᾽ οὐ Καλλίου ἀνθρώπου): πάλιν ἐν τούτοις ἵσταται, ἕως ἂν τὰ ἀμερῆ στῇ καὶ τὰ καθόλου, οἷον τοιονδὶ ζῷον, ἕως ζῷον, καὶ ἐν τούτῳ ὡσαύτως. δῆλον δὴ ὅτι ἡμῖν τὰ πρῶτα ἐπαγωγῇ γνωρίζειν ἀναγκαῖον: καὶ γὰρ ἡ αἴσθησις οὕτω τὸ καθόλου ἐμποιεῖ.[131]

It is noticeable that, here, Aristotle does not mention abstraction at all in accounting for the process of general concept acquisition. Another important point is that Aristotle maintains that there is (in a sense) perception of universals.[132] This is a quite different position from that of other abstractionists. Since they adhere to an ontology of particulars,

[131] *Apo*., II, 19, 100a15-100b5.

[132] The claim that universals can be objects of perception would seem to come into conflict with another claim of Aristotle's that perception is of particulars (*Apo*., I, 31). On this issue, Barnes argues that we perceive particular things as *F*-things. Namely, perception involves implicit recognition of a universal*, F,* although we shall not have explicit knowledge of *F* until we advance to the stage of concept-acquisition. See Barnes (1994) p. 266.

For the process of acquiring knowledge of universals, see *Apo.*, II, 19, 99b34-100a10. There is a close similarity between this passage and *Met.*, I, 1, 980b27-981a3. For Barnes's interpretation of Aristotle's process of the acquisition of universals, see Barnes (1994) pp. 262-265.

i.e., "all things that exist are only particulars,"[133] universals or general ideas have to be invented by the mind, and abstraction is an epistemic apparatus by which universals are produced. Thus, Locke says,

> General and Universal, belong not to the real existence of Things; but are the Invention and Creatures of the Understanding, made by it for its own use, and concern only Signs, whether Words, or Ideas. Words are general, as has been said, when used, for Signs of general Ideas; and so are applicable indifferently, when used, for Signs of general Ideas; and so are applicable indifferently to many particular Things; And Ideas are general, when they are set up, as the Representatives of many particular Things: But universality belongs not to things themselves, which are all of them particular in their Existence, even those Words, and Ideas, which in their signification, are general.[134]

But, for Aristotle, since universals exist independently of our mind, in order to have a general concept, we need not invent it, we need only have the requisite capacity for grasping it. This obviates for Aristotle's account of concept-acquisition any need for any such mental operation as epistemic abstraction. If Aristotle does not require such an epistemic apparatus as the nominalists' abstraction, it is fairly reasonable to infer that his sort of abstraction must be something different from that of the nominalists.

If there is anything analogous to epistemic abstraction in Aristotle, it would be induction rather than abstraction. On this point, we can examine another passage interpreted by some commentators to mean that Aristotle holds some version of abstraction to be involved in grasping a universal:

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[133] Locke (1975) BK. Ch. III, 3, §6.
[134] *Ibid.,* BK. Ch. III, 3, §11.

> And it is necessary to seek first what they all have in common, looking at things which are similar and undifferentiated. Then [do the same thing] again for other things which are in the same kind as the first group and are of the same form as one another but of a different form from the first group. When we have grasped what is common among all these, and similarly for the other groups, we must investigate again whether those things you have grasped have anything in common, until you come to a single account; for this will be the definition of the object. If you reach not a single account but at two or more, then it is plain that what you are seeking is not one item but several.
>
> Ζητεῖν δὲ δεῖ ἐπιβλέποντα ἐπὶ τὰ ὅμοια καὶ ἀδιάφορα, πρῶτον τί ἅπαντα ταὐτὸν ἔχουσιν, εἶτα πάλιν ἐφ᾽ ἑτέροις, ἃ ἐν ταὐτῷ μὲν γένει ἐκείνοις, εἰσὶ δὲ αὑτοῖς μὲν ταὐτὰ τῷ εἴδει, ἐκείνων δ᾽ ἕτερα. ὅταν δ᾽ ἐπὶ τούτων ληφθῇ τί πάντα ταὐτόν, καὶ ἐπὶ τῶν ἄλλων ὁμοίως, ἐπὶ τῶν εἰλημμένων πάλιν σκοπεῖν εἰ ταὐτόν, ἕως ἂν εἰς ἕνα ἔλθῃ λόγον: οὗτος γὰρ ἔσται τοῦ πράγματος ὁρισμός. ἐὰν δὲ μὴ βαδίζῃ εἰς ἕνα ἀλλ᾽ εἰς δύο ἢ πλείους, δῆλον ὅτι οὐκ ἂν εἴη ἕν τι εἶναι τὸ ζητούμενον, ἀλλὰ πλείω.[135]

Barnes interprets this passage as describing a method of definition by abstraction.[136] Obviously, the passage is closely reminiscent of Locke's explanation of the process of obtaining a general idea by abstraction. Locke writes:

> And thus they come to have a general Name, and a general idea. Wherein they make nothing new, but only leave out of the complex Idea they had of Peter and James, Mary and Jane, that which is peculiar to each, retain only what is common to them all.[137]

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[135] *Apo.*, II, 13, 97b7-16.
[136] Barnes (1994) pp. 248-249.
[137] Locke (1975) BK. III, 3, §7.

While Aristotle is describing a way of defining a general term, Locke is describing the process of obtaining a general concept; nevertheless, in both passages, defining a general term and obtaining a general concept seem to be characterized as involving the same procedure, namely, leaving out something different and retaining something common among particulars. If this is so, we have an indication that Aristotle has the same, or a similar, epistemic method to the nominalists' abstraction.

However, in order to determine whether the processes as presented in the two passages are exactly the same, we must address ourselves to the following two questions. The first is what Locke means by 'what is common to them all'; in the case of Aristotle, this phrase would be parsed as a universal. But, because of his ontology, Locke cannot maintain that any common element among particulars exists. For Locke, a general concept refers to a general idea which has its genesis and is held purely in our mind. Thus, there is good reason to resist the identification of the epistemic process described in the first passage with that of the second. The second question concerns whether Aristotle would label the process involved in the definitional method as 'abstraction'. Even if the two processes are the same, it could be the case that the process Aristotle is talking about in the first passage is something other than his sort of abstraction.

There are in fact several reasons to think that this process is not Aristotle's abstraction. First, he does not use the term 'abstraction (ἀφαίρεσις)' at all in the passage; secondly, abstraction in his sense is generally applied to a single particular thing like a man, or horse, rather than to a group of particulars sharing a certain property, other than in the case of selecting by abstraction a numerical property from a group of things; and thirdly, in most passages where Aristotle mentions abstraction, the term appears alongside 'separation (χωρισμός)' or '*qua (ᾗ)*', while in this passage these terms are noticeable by their absence. Finally, as we already have seen, for Aristotle, induction is the process through which a universal is acquired from particulars. So, it may be the case that what Aristotle describes in the passage is not definition by abstraction but definition by induction or inductive definition. The next passage confirms this inference:

> Having made these distinctions, we must distinguish how many forms of dialectical arguments there are. There is induction as well as deduction. And what deduction is has been said before. And induction is the progress from the particulars to the universal. For instance, if the one who has knowledge is the best pilot and the best charioteer, then in general the one who knows is best in each [area].
>
> Διωρισμένων δὲ τούτων χρὴ διελέσθαι πόσα τῶν λόγων εἴδη τῶν διαλεκτικῶν. ἔστι δὲ τὸ μὲν ἐπαγωγή, τὸ δὲ συλλογισμός. καὶ συλλογισμὸς μὲν τί ἐστιν, εἴρηται πρότερον. ἐπαγωγὴ δὲ ἡ ἀπὸ τῶν καθ᾽ ἕκαστα ἐπὶ τὸ καθόλου ἔφοδος: οἷον εἰ ἔστι κυβερνήτης ὁ ἐπιστάμενος κράτιστος, καὶ ἡνίοχος, καὶ ὅλως ἐστὶν ὁ ἐπιστάμενος περὶ ἕκαστον ἄριστος.[138]

This passage is unequivocal that it is induction that makes the transition from particulars to a universal; whereas Aristotle never explicitly says that abstraction is necessary for acquiring universals. Furthermore, the example given of induction shows that induction, as much as epistemic abstraction, involves taking what is common among particular

138 *Top.*, I, 12, 105a10-16.

instances. So we have every indication that the first passage is concerned not with explicating definition by abstraction but rather definition by induction.

6. Abstraction as a Mental Operation.

Our final area of enquiry in this chapter will be the idea that Aristotle's abstraction is a mental operation. This idea retains some currency among modern commentators; for instance, Annas considers Aristotle's abstraction to be an act of ignorance (i.e., willed inattention),[139] and Mignucci argues that abstraction is a mental activity at any rate.[140]

Before examining the relevant texts, it is appropriate first to clarify the meaning of 'mental'. I will confine the meaning of 'mental' to subjective inner private experience, such as the experience of *qualia*, pain, ignoring, paying attention, etc. In this narrow sense, I would argue that Aristotelian abstraction is far from being mental. In the case of Frege, for instance, abstraction refers to a deliberate lack of attention.[141] In these contexts it is legitimate for him to call any attempt to found arithmetic on the basis of abstraction psychologism, or a subjectivism in the philosophy of mathematics.

There are several places which seem to imply that Aristotle's abstraction is a mental activity in the narrow sense. The following is one:

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139 Annas (1976) p29.

140 Mignucci (1987) p. 181.

141 Lear (1982) p.185; Frege (1952) pp. 84-85.

One thinks of *those things which are spoken of as in abstraction* just as, if one thought of the snub, not *qua* snub, but actually separately *qua* concave, one would think of it apart from the flesh in which the concave [exists]. One thinks of mathematical objects which are not separate, as separate, when one thinks of them.

τὰ δὲ ἐν ἀφαιρέσει λεγόμενα <νοεῖ> ὥσπερ ἂν εἴ <τις> τὸ σιμὸν ᾗ μὲν σιμὸν οὔ, κεχωρισμένως δὲ ᾗ κοῖλον [εἴ τις] ἐνόει [ἐνεργείᾳ], ἄνευ τῆς σαρκὸς ἂν ἐνόει ἐν ᾗ τὸ κοῖλονοὕτω τὰ μαθηματικά, οὐ κεχωρισμένα <ὄντα>, ὡς κεχωρισμένα νοεῖ, ὅταν νοῇ <ᾗ> ἐκεῖνα. [142]

It is tempting to read the first sentence as saying that to abstract the figure of snub is to conceive of snub, not *qua* snub, but *qua* concave. This reading tends to buttress a view of abstraction as a mental act of ignoring of a private or subjective character, because to conceive of snub only *qua* concave is to pay attention only to relevant aspects of snub and to ignore other, circumstantial aspects. Equally, though, the sentence could be also read as meaning that we can think of the concavity without considering other elements of the snub by abstracting the concavity from the snub. On this reading, it is one thing to separate some aspect of an object by abstraction, and another thing to think of that separated aspect. To clarify this point, let us consider the following sentence:

(i) We conceive of the snub *qua* concave.

[142] *DA.*, III, 7, 431b12-431b17.

In my analysis of Aristotle's abstraction, this sentence means that we conceive only *kath' hauta* properties of concavity among the properties of a snub. What abstraction engages in is the process of selecting or separating those *kath' hauta* properties from other properties; abstraction has nothing to do with the mental act of 'conceiving'. Suppose that X represents a verb, Y, a sensible object, and Z a predicate of Y, respectively. Then, (i) can be formulated more generally as follows:

(ii) We X Y *qua* Z.

I argued earlier that '*qua*' can seen as a linguistic function mapping an ordered pair, <Y, Z>, to a set of *kath' hauta* properties of Z which also belong to Y;[143] when we identify abstraction with a method of selecting a group of predicates, it is obvious that X does not get involved in the process of abstraction. For the function, *qua*, what determines the value of *qua* are the values of Y and Z; X does not change the value of *qua* at all. We can replace X with a verb such as 'conceive,' 'study,' 'investigate,' etc., but whatever verb we choose, the verb does not partake of the process of abstraction. In the case of (i), due to abstraction, i.e., by (conceptually or linguistically) separating *kath' hauta* properties of concavity from the snub by abstraction, we can think only of the concavity without considering material aspects of the snub. But thinking (only the concativity) is not abstraction; abstraction is the process of selecting (picking out) those *kath' hauta* properties of concavity.

[143] See Chapter Two, §3.

Greek grammar also supports this interpretation. In the text, 'τὰ ἐν ἀφαιρέσει λεγόμενα' is the grammatical object of 'νοεῖ'; and it is natural to regard the phrase 'τὸ σιμὸν ᾗ μὲν σιμὸν οὔ, κεχωρισμένως δὲ ᾗ κοῖλον' in the sentence after 'ὥσπερ ἄν' as an example of 'τὰ ἐν ἀφαιρέσει λεγόμενα'. Thus, the above passage does not equate thinking with abstraction; it only explains by taking a concrete example what it would be like to think of 'a thing which is separated by abstraction': thinking of 'a thing separated abstraction' is like thinking of 'the snub *qua* concave'. That is, the snub *qua* concave is an example of 'τὰ ἐν ἀφαιρέσει λεγόμενα'[144] Further, the passage offers no more information on how abstraction is accomplished; we have only an example of abstracted things.

There is another reason to prefer the second reading. Suppose that the first reading is right. This would identify abstraction with conceiving or thinking generally. In *De Anima*, III, Aristotle uses 'thinking (νοεῖν)' in two different senses. In a narrow sense, 'thinking (νοεῖν)' means knowing for Aristotle, rather than merely 'thinking about things'. But the term is also used in such a general way that it could refer to any intellectual activity such as reasoning, conceptual analysis, paying attention, etc.[145] It is not certain whether Aristotle uses 'νόησις' (thinking or thought) technically in the narrow sense or in the broader sense in the above passage. But the second reading is

[144] Another similar expression is 'τὰ ὄντα ἐν ἀφαιρέσει' (*DA*., 4, III, 429b18).

[145] I will discuss again Aristotle's two uses of 'thinking (νοεῖν)' in *De Anima*, III. See Chapter Four, §4.

constistent with whichever sense he may have intended. Let us consider each case separately.

Suppose that, in the above passage, Aristotle uses 'think' not technically but in a broad sense. Then, this passage simply does not provide us with enough information to decide whether abstraction is an inner private mental process or not; because thinking in such a broad sense may include not only private mental operations, but also publicly communicable intellectual activities such as mathematical calculation, logical inference, conceptual analysis, etc.

Now, suppose that he uses the term technically. In Aristotle's technical usage, 'νόησις' is frequently used in contrast with perception (αἴσθησις).[146] Thinking is distinguished from perceiving in that: (i) only a few animals have the capacity of thinking, whereas all animals are capable of perceiving.[147] (ii) Thought is fallible but the perception of proper objects (ἡ αἴσθησις τῶν ἰδίων) always true or liable to falsity to a minimal degree.[148] But, to emphasize the passivity of thinking, Aristotle also likens thinking to perceiving: to think is to receive the forms of objects of thought, just as perception is to receive the forms of perceptible objects;[149] thinking as the process of receiving the form is passive; because to think is to become (in a sense) the object of the

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[146] Peters (1967) p. 124.
[147] *DA.*, III, 3, 427b7-8.
[148] *Ibid.*, 427b8-13.
[149] *Ibid.*, 4, 429a13-19.

thinking by receiving its form.[150] It is difficult to equate thinking in this sense with such an active mental operation as abstraction. Abstraction—whether it is a mental activity such as generalization, paying attention to something, disregarding, or is conceptual analysis—involves very active operations, in which the mind, the subject of abstraction, always manipulates or modifies the objects of abstraction; whereas, in thinking (in the narrow sense), the mind is not comparably active; for Aristotle, thinking is a passive process rather than an active one.

Those holding to a view of Aristotelian abstraction as a mental process might, however, object that, for Aristotle, thinking is not quite entirely passive. In fact, Aristotle argues that the intellect also has an active role in thinking:

> Since just as in the whole of nature there is some matter to each genus (and this is potentially all of them), but its cause is different and productive in the sense that it makes all things, e.g., an art is related to its material in this way, and these differences must be also in the soul. And there is such an intellect in the sense that it becomes all things, and there is another intellect in the sense that it produces all things, as a kind of disposition, like light, does; for in a way light also makes colors in potentiality colors in actuality.
>
> ᾽Επεὶ δ᾽ [ὥσπερ] ἐν ἁπάσῃ τῇ φύσει ἐστὶ [τι] τὸ μὲν ὕλη ἑκάστῳ γένει (τοῦτο δὲ ὃ πάντα δυνάμει ἐκεῖνα), ἕτερον δὲ τὸ αἴτιον καὶ ποιητικόν, τῷ ποιεῖν πάντα, οἷον ἡ τέχνη πρὸς τὴν ὕλην πέπονθεν, ἀνάγκη καὶ ἐν τῇ ψυχῇ ὑπάρχειν ταύτας τὰς διαφοράς: καὶ ἔστιν ὁ μὲν τοιοῦτος νοῦς τῷ πάντα γίνεσθαι, ὁ δὲ τῷ πάντα ποιεῖν, ὡς ἕξις τις, οἷον

[150] *Ibid.*, 429b3-4; 8, 431b26-432a3. The identity between thinking and its objects will be discussed again at Chapter Four, §4.

τὸ φῶς: τρόπον γάρ τινα καὶ τὸ φῶς ποιεῖ τὰ δυνάμει ὄντα χρώματα ἐνεργείᾳ χρώματα.[151]

This is a much-discussed passage in Aristotle scholarship, on the basis of which Thomas drew a distinction between the passive and the active intellects, and also inferred that the active intellect abstracts the species from *phantasmata* (φαντάσματα) produced by sense and imposes this on the passive intellect.[152] Thomas seems to argue in the following way:

(1) We can think of something by having its *phantasma* .[153]

(2) The *phantasma* derive from sense perception.[154]

(3) But, since we can think of only the form of an object,[155] it is necessary to separate the pure intellectual form from the sensual elements of the object.

(4) Such separation is possible by abstraction.

[151] Ibid., 5, 430a10-17.

[152] This abstraction corresponds to the second abstraction of the two kinds of abstractions Thomas distinguishes in his commentary on *Metaphysics*, where he interprets Aristotle's abstraction as an epistemological process deriving the acquisition of universals. Thomas argues that a form, which is a universal, is abstracted from its matter (Chapter Four, §2, n. 286). See Hamlyn (1968), p. 141.

[153] *DA*., III, 7, 431a14-15; 431b2; 8, 432a3-6.

[154] *Ibid*., 8, 432a3-4.

[155] *Ibid*., 431b26-432a1.

But, despite Thomas' reading, what the passage above tells us is no more than that there must be a kind of cause in the mind, which brings about the actualization of its potentialities; and it is hard to connect such a role with abstraction. There is no textual evidence that Aristotle thinks of abstraction as a necessary condition for grasping intellectual forms. Rather, Aristotle is inclined to think that, in principle, each epistemological apparatus grasps its proper objects; i.e., the senses perceive sensible forms without the matter such as color, sound, or flavor,[156] and the intellect grasps the intelligible form of an object such as the rationality of a man. If these claims are correct, it does not fall to Aristotle to explain how the intellect grasps a form separately from its *phantasma*; because the intellect only grasps the form. If Aristotle does not assume any cognitive process to separate a sensible form from a sensible object, there would be no need for him to posit a special apparatus to separate an intelligible form from a *phantasma*, either. Moreover, Aristotle does not seem to believe that it is necessary to separate a form from its *phantasma* in order to conceive any object. It is rather the other way around: We always think with *phantasmata*[157] and thinking is not possible without *phantasmata*. In fact, as we have seen, separation by abstraction has nothing to do with such epistemic processes as perceiving or knowing. Abstraction, on the view of this chapter, instead consists of a logical analysis in which predicates are separated from their subject.

[156] *Ibid*., 12, 424a17-24. But there are also common sensibles like movement, stasis, number, figure, or size, which are perceived by more than one sense. See *Ibid*., II, 6, 418a17-20.
[157] *Ibid*., III, 7, 431b2-4.

However, according to Aristotle, the intellect has another kind of active function: When intellect operates through judgment it also combines and divides concepts.[158] In this sense, abstraction might be said to be a certain way of thinking; we saw that abstraction involved a kind of conceptual separation. There are two ways to respond to this redefinition of abstraction as thinking. First, if thinking includes all forms of conceptual analysis, there is no need to take issue with the idea that abstraction is a sort of thinking. My objection to the traditional line was that Aristotle's abstraction can neither be identified with private mental activities nor with thinking in the technical sense, i.e., grasping a form of an object. Secondly, even if division (διαίρεσις) by thought is a kind of conceptual analysis and is performed by thought, it is trivially true that not every conceptual analysis will be a form of abstraction. By 'διάιρεσις,' Aristotle usually means a method of definition by genus and differentiae. So it is safe to say that there is no guarantee that the division of a concept by the intellect is necessarily what Aristotle was getting at in his term 'abstraction'.

The objection to the idea that Aristotle's abstraction is a kind of thinking is applicable also to mathematical cases:

> Now the mathematician also treats of these [sensible] things, but not as boundaries of each natural body; nor does he consider the properties as if it belongs to such bodies. That is why he separates [them]; for in thought they are separable from motion, and it makes no difference, nor does any falsity result, even if they are separated.

[158] *Top.*, VI, 143a29-145b34.

> περὶ τούτων μὲν οὖν πραγματεύεται καὶ ὁ μαθηματικός, ἀλλ' οὐχ ᾗ φυσικοῦ σώματος πέρας ἕκαστον: οὐδὲ τὰ συμβεβηκότα θεωρεῖ ᾗ τοιούτοις οὖσι συμβέβηκεν: διὸ καὶ χωρίζει: χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι, καὶ οὐδὲν διαφέρει, οὐδὲ γίγνεται ψεῦδος χωριζόντων.[159]

As Cleary points out,[160] "χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι" can be translated in two different ways. If we may translate 'χωριστά' as 'separated' and take 'τῇ νοήσει' to be an instrumental dative, the sentence will be rendered as 'they are separated by thought'. But, if we translate 'χωριστά' as 'separable' and take 'τῇ νοήσει' to be a locative dative, it translates as 'they are separable in thought.' Many commentators have adopted the first translation, the implication of which is that Aristotle's abstraction is a mental activity just to the extent that thought is. For instance, referring to the passage, Mignucci argues that 'the presence of a mental activity in the determination of the object of mathematics is confirmed' in the text, because there 'Aristotle says that mathematical objects are separated by a mental act from sensible things.'[161] But, as I have already argued, considering some property of a sensible particular as being separated from the sensible is one thing, and actively separating the property from the sensible another. So, the second translation seems to be more plausible. Then, what the sentence means is simply that after abstracting certain properties of

[159] *Phy.*, II, 2, 193b31-35.
[160] Cleary (1985) p.34.
[161] Mignucci (1987) p. 181.

sensible things, we can consider or treat them as separate entities—though they are not separated in reality.

Chapter Three

Problems in Aristotle's Philosophy of Mathematics

1. Aristotle's Naïve Mathematical Realism

I made the claim earlier that mathematics is the best case supporting AFS. Thus, to rebut AFS, it is important for Aristotle to show that his theory of abstraction also applies to mathematics and mathematical entities. Aristotle obviously seems to believe that he can have recourse to abstraction to explain how it is that mathematical objects are not separated from sensible objects. It is relatively frequent for him to use mathematical examples in order to illustrate abstraction and to express mathematical objects in the terms of abstraction. For example, mathematical objects are referred to as "things said as a result of abstraction (τὰ ἐξ ἀφαιρέσεως λεγόμενα)" "things spoken of in abstraction (τὰ ἐν ἀφαιρέσει λεγόμενα)" or "things obtained through abstraction (τα δι' ἀφαιρέσεως)" etc.[162]

It has been a problem for commentators that Aristotle expresses mathematicals not as 'things which are abstracted' but as 'something said as a result of abstraction (τὰ ἐξ αφαιρέσεως λεγόμενα)'.[163] An interpretation of this phrasing is possible, however, that further backs up my analysis of the nature of Aristotle's abstraction.

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[162] Bonitz (1870) 126b21-26. We do not have textual information as to what Aristotle means exactly by those expressions.

[163] See Mueller (1970) p.159.

Suppose, for example, that geometry studies a sensible object *qua* triangle. Then, according to the line of argument put forward in the last chapter, geometry would first decide by abstraction which properties of the sensible object primitively belong to a triangle. In that sense, we may say that what geometry is concerned with is not triangularity itself, but, rather the primitive properties of triangles. When geometry studies something as a triangle, it does not ask what a triangle is; it asks what properties of that thing universally belong to triangles (the theorems relating to triangles can be regarded as answers to this question); furthermore, we identify these properties by *abstracting or removing* (ἀφαιρεῖν), in order of increasing generality, other predicates than 'being a triangle' from the object. Therefore, it is more appropriate to say that the objects of geometry are obtained after abstraction or as a result of abstraction, than that they are abstracted; when we study a sensible object *qua* triangle, what we abstract (or remove) from that object are properties of the object in a certain order, and as the result of such abstraction, we obtain the *kath' hauta* properties of a triangle which belong to the object. In that sense, the objects of mathematics are "things said as a result of abstraction (τὰ ἐξ ἀφαιρέσεως λεγόμενα)," "things spoken of in abstraction (τὰ ἐν ἀφαιρέσει λεγόμενα)," or "things obtained through abstraction (τα δι' ἀφαιρέσεως)," etc.[164]

[164] Many commentators have been convinced that those phrases exclusively refer to mathematical objects. For this point, see, Cleary (1985) p. 16. But if my interpretation of abstraction is correct, Aristotle's abstraction is devised as a solution to the problem of complexity on which ASF relies. Since ASF is applied not only to mathematical but to all sciences, abstraction should not be confined to mathematical objects. In fact, it is not difficult to find Aristotle employing abstraction

One might object at this point that geometry studies not only *kath' hauta* properties of triangle, but also triangularity itself; geometry gives the definition of a triangle, namely, triangularity or a triangle is a mathematical object. But triangularity is also known as an object of geometry *as a result of the abstraction* of other predicates than 'being a figure' from a sensible object. When a geometer studies a triangular sensible object *qua* figure, he comes to know that its triangularity is a *kath' hauto* property of a figure by abstracting the predicates of the sensible object in a proper order. What is abstracted is, then, not 'being a triangle' but the sensible object's predicates other than 'being a figure' Thus it still makes a sense to say that mathematical objects are things spoken of as a result of abstraction (τὰ ἐξ ἀφαιρέσεως λεγόμενα).

I have argued that, when A studies B *qua* C, the objects of A are nothing but certain properties of B. Since for Aristotle, arithmetic is a study of a sensible particular *qua* individual, and geometry studies such an individual *qua* line, plane, or solid,[165] it should be the case that mathematics consists of a study of the properties of sensible particulars. Thus, if abstraction serves as a method of obtaining mathematical objects, Aristotle's view on mathematics should amount to a kind of naïve realism, the view that mathematical objects exist as properties of sensible particulars.

The naïve realistic interpretation of Aristotle's philosophy mathematics seems also to be confirmed by the ontology developed in *Categories*. In *Categories*, quantity

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to explain the objects of other sciences. For instance, at *Met.*, XIII, 3, 1077b17-1078a21, abstraction is suggested as a general way to delimit the domain of each science, e.g., the science that has the healthy as its subject treats things *qua* healthy, just as geometry treats things *qua* line, plane or solid.

[165] *Met.*, XIII, 3, 1078a25-27.

appears as one of ten categories; Aristotle specifically divides quantities into two kinds: discrete and continuous,[166] depending on whether or not a quantity has a common boundary at which its parts join together. For example, in the case of lines, a point on a line can mark such a boundary. The genus of continuous quantities includes geometrical objects such as lines and surfaces, as well as time and space. Discrete quantities have members including number and language, the former of which is the object of arithmetic.[167] So, from the perspective of the ontology of *Categories*, mathematics is a science of quantity, which is in turn a category of the properties of individual substances.[168]

Aristotle's categories are not only a list of kinds of predicates, but also a list of kinds of items in his ontology.[169] Thus, it can be said that Aristotle takes mathematical objects to be a kind of beings in his ontology. Aristotle also argues that all the other categorical beings ontologically depend on individual substances;[170] and since the former

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[166] He also divides quantity into plurality and magnitude. The former is numerable and the latter measurable. This division exactly corresponds to the division between discrete and continuous quantities. Continuous quantity corresponds to the common sensible of magnitude, and discrete quantity to the common sensible of number. Aristotle explains that while plurality is potentially divisible into non-continuous parts, magnitude is divisible into continuous parts. He includes line, breadth, and surface under magnitude; number under plurality. *Cat*., 6, 4b20.

[167] *Ibid.*, 6. 4b22-5a14.

[168] This is the view endorsed by Apostle, who sees Aristotle as construing mathematics as a science of the quantities of sensible objects. See Apostle (1952) p.14 and 18..

[169] Michael Frede distinguishes two senses of 'category': 'category' means (i) 'predication' as in *Topics,* and (ii) 'kinds of things' as in *Categories.* See Frede (1981) p. 35. For another interpretation of the different senses of 'category', see also Morrison (1992) n. 3.

[170] "Thus all other things are either said of the primary substances as subjects or in them as subjects. So, if the primary substance did not exist it would be impossible for any of the other things to exist (ὥστε τὰ ἄλλα πάντα ἤτοι καθ᾽ ὑποκειμένων τῶν πρώτων οὐσιῶν

are predicated of the latter, but not *vice versa*, the former can be regarded as the properties (συμβεβηκότα) of the latter. Thus, the fact that the objects of mathematics belong to the category of quantity implies that (i) the existence of mathematical objects depends on the existence of individual substances, and (ii) that mathematical objects exist as properties of individual substances.

This claim of a naïve realism in mathematics finds further confirmation in Aristotle's theory of perception. In *De Anima*, Aristotle distinguishes common sensible objects from proper sensible objects. Aristotle argues that some objects of sense have their own proper sense faculties; for instance, color is sensed only by sight, and sound only by hearing, etc. Other objects of sense, such as movement, rest, magnitude, number, figure and size, are not special to any sense faculty, but common to more than one. For instance, movements are perceptible by both touch and sight.[171]

It should be noted that Aristotle includes both magnitude and number among common sensible objects, each of which is the proper object of geometry and arithmetic respectively. Thus, it is fairly reasonable to infer that Aristotle takes mathematical objects

λέγεται ἢ ἐν ὑποκειμέναις αὐταῖς ἐστίν. μὴ οὐσῶν οὖν τῶν πρώτων οὐσιῶν ἀδύνατον τῶν ἄλλων τι εἶναι) (*Cat*., 5, 2b4-6)." Aristotle takes the individual man or individual horse as examples of primary substances (*Ibid.,* 2a11-15). In *Categories*, a substance "is neither said of a subject nor in a subject (ἣ μήτε καθ' ὑποκειμένου τινὸς λέγεται μήτε ἐν ὑποκειμένῳ τινί ἐστιν) (*Ibid*., 2a11-13)." All other things are either said of the primary substances as subjects or in them as subjects. But the species of the primary substances and genera of those species are also called the secondary substances in that other things i.e. things except the primary substances are predicated of them. (*Ibid*., 2a19-3a5)

[171] *DA*., II, 6, 418a18-19. Another passage worth looking at is *Ibid*., III, 1, 425a14-27, where Aristotle not only mentions number and geometrical properties as common sensibles, but also argues why there cannot be a special sense faculty for common sensibles.

to be properties of sensible objects. Moreover, since mathematical objects are not only properties of individual objects but also perceivable according to his theory of perception, such objects turn out to be objects of sense experience.[172] In that sense, Aristotle's view of mathematics also deserves the title of some form of mathematical empiricism.

2. The General Theory of Proportion

Aristotle's interpretation of the general theory of proportion can be taken as further evidence of his mathematical naïve realism. Aristotle mentions this general theory of proportion in several places, mostly for the purpose of arguing for the mathematicals' inseparability from sensible particulars. Aristotle holds that there is a universal mathematics which can be applicable to different kinds of mathematical beings.[173]

It is generally agreed that the universal mathematics to which Aristotle is referring is Eudoxus' general theory of proportion.[174] Aristotle contrasts this 'universal' mathematics with other mathematical sciences in suggesting that the first is applicable indifferently to any kind of quantity—that is, to numbers, lines, solids, or times—while other mathematical sciences deal with only a certain particular kind of quantity, e.g. geometry studies lines, figures and solids, and arithmetic numbers.[175]

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[172] Corkum agrees on this point. See Corkum (2004) n. 5.
[173] *Apo*., I, 5, 74a18-55; 9, 76a22-25; II, 17, 99a9-11; *Met.*, IV, 1, 1004a6-10; VI, 1, 1026a24-27; XIII, 3, 1077b17-20; 2, 1077a9-14; XI, 7, 1064b6-9, etc. See also Heath (1949) pp. 43 and 223.
[174] cf. Ross (1924) *Vol. II,* p. 415.
[175] *Apo*., I, 5, 74a18-25.

This feature of universal mathematics is also the distinctive feature of the general theory of proportion as presented in *Elements*, Book V. The universal applicability of the theory in Book V, distinguishes this theory from another theory of proportion given in Book VII; while the former applies to all magnitudes and numbers, the latter holds good only of integral numbers.[176] Aristotle's illustration of the general theory of proportion also confirms that the universal mathematics he cites is the theorem presented in *Elements*: namely, that proportions alternate (if a:b::c:d, then a:c::b:d), which Aristotle takes as an example of a universal mathematical proposition, corresponds to proposition 16 of Book V in *Elements*.[177] On these grounds there can be little doubt by universal mathematics Aristotle means Eudoxus' theory of proportion.

From the universal applicability of the general theory of proportion, Aristotle tries to draw the implication that mathematics is a study of sensible individual things. Aristotle says,

> Just as the universal part of mathematics is not concerned with objects which exist separately apart from magnitudes and numbers, but is concerned with these [magnitudes and numbers], but not *qua* such as to have magnitude or to be divisible, it is evident that there can be both formulae and demonstrations about sensible magnitudes, but not *qua* sensible but not *qua* having certain definite qualities.
>
> ὥσπερ γὰρ καὶ τὰ καθόλου ἐν τοῖς μαθήμασιν οὐ περὶ κεχωρισμένων ἐστὶ παρὰ τὰ μεγέθη καὶ τοὺς ἀριθμοὺς

[176] The theory of proportion in Book VII is regarded as the older theory, which was developed by Pythagoreans before Eudoxus. See Heath (1949) p. 43.

[177] *Elements*, V, 16. Aristotle's own proof of this proposition is found at *Apo*., I, 6, 74a24. His proof is slightly different from Euclid's; the former is more algebraic. See Lear (1982) p. 166.

> ἀλλὰ περὶ τούτων μέν, οὐχ ᾗ δὲ τοιαῦτα οἷα ἔχειν μέγεθος ἢ εἶναι διαιρετά, δῆλον ὅτι ἐνδέχεται καὶ περὶ τῶν αἰσθητῶν μεγεθῶν εἶναι καὶ λόγους καὶ ἀποδείξεις, μὴ ᾗ δὲ αἰσθητὰ ἀλλ᾽ ᾗ τοιαδί.[178]

Aristotle's argument is that since the objects of the general theory of proportion are not separated from numbers and magnitudes, it is *possible* that the objects of geometry or arithmetic are not separated from sensible objects.

A couple of questions can be raised: First, what is the rationale for thinking that the objects of the general theory of proportion, namely, proportions, are not separated from numbers and magnitudes? Second, does it follow from the fact that the objects of the general theory of proportion are not separated from numbers and magnitudes that other mathematical objects are not separate from sensible objects? Let us consider the first question first.

Aristotle's own answer to the first point is that it is absurd to posit a proportion as a separate substance. But why is it absurd? What Aristotle has in mind seems to be the problem of multiplication:

> Again, there are some general propositions stated by mathematicians, extending beyond these substances. Then, there will be another substance between, and separate from, the Forms and the intermediates, which is neither number nor points nor spatial magnitude nor time. And if this is impossible, it is obvious that the former substances cannot exist apart from sensible objects.

[178] *Met.*, XIII, 3, 1077b17-22.

> ἔτι γράφεται ἔνια καθόλου ὑπὸ τῶν μαθηματικῶν παρὰ ταύτας τὰς οὐσίας. ἔσται οὖν καὶ αὕτη τις ἄλλη οὐσία μεταξὺ κεχωρισμένη τῶν τ᾽ ἰδεῶν καὶ τῶν μεταξύ, ἣ οὔτε ἀριθμός ἐστιν οὔτε στιγμαὶ οὔτε μέγεθος οὔτε χρόνος. εἰ δὲ τοῦτο ἀδύνατον, δῆλον ὅτι κἀκεῖνα ἀδύνατον εἶναι κεχωρισμένα τῶν αἰσθητῶν.[179]

According to AFS, there must be Forms of all things of which there are sciences.[180] So, for Platonists, there must be also Forms of the propositions that make up the objects of the general theory of proportion. However, since Platonists differentiate mathematicals from Forms and categorize the former as intermediates, proportions should be identified with some intermediates. Aristotle counters at this point that a proportion cannot belong to the domain either of the Forms or the intermediates, but must exist as another substance between them. Given that Platonists distinguished mathematical objects from Forms on account of the former's lack of uniqueness, it would seem reasonable to posit that such Platonists cannot identify a proportion with a Form. But why should it not be classed as an intermediate? Clearly, a proportion between two numbers, for instance, is different from those numbers themselves; and equally the proportion of two lines is not coincident with those lines. Thus, it could be said that a proportion cannot be identified with any of the intermediates to which it is applied.

That does not give us, however, a sufficient reason to exclude proportions from the domain of intermediates. It is accepted that intermediates are constituted of different

[179] *Ibid.*, 2, 1077a9-14.
[180] *Ibid.*, 1079a7-8.

kinds of mathematical objects; for example, of numbers as well as geometrical objects such as figures or lines. But they all belong to the same domain, insofar as they are dealt with in mathematics. So, Platonists could simply add proportions to the realm of the intermediate as another kind of mathematical objects. Thus, unless the difference between a proportion and an intermediate is such as to place each of them on different ontological levels, there is no ground for saying that a proportion belongs to another domain than that of the intermediates.

Someone might try to base such an ontological difference on the fact that a proportion is not a monadic property. If a proportion can be seen as a kind of relation between two mathematical objects, and a relation is to be distinguished from its terms, a proportion also should be distinct from the mathematical objects it relates. Problematically, though, such a distinction between a relation and a monadic property would have been unfamiliar to both Plato and Aristotle.[181] In *Phaedo*, for instance, Plato posits Forms of relations such as 'being equal' or 'being tall' along with Forms of monadic properties such as 'being beautiful'. Aristotle also seems to allow the same ontological status to relations and to monadic properties alike; for him, both quality and

[181] Some modern philosophers distinguish relations from monadic properties. First, relations contain at least two terms, that is, relations defined as shared by more than one individual. Second, some temporal or spatial relations cannot be possessed in common by more than one individual, while most monadic properties may be shared by many individuals. For discussion of the difference between relations and monadic properties, see Russell (1992) p. 274 and O'Connor (1976) p. 279.

relation are equally dependent on individual substances in terms of existence; they are all equally the properties of individual substances.[182]

It might be argued that a proportion is always a proportion *of* something, namely, a certain kind of mathematical quantity; so that a proportion cannot exist independently of mathematical objects such as number or magnitudes. But that does not provide a reason for Platonists to accept the inseparability of a proportion from mathematical quantities. Although a property of being beautiful, for instance, is always a property of something such as a man or flower, Platonists do not think for that reason that Beauty itself is inseparable from whatever instantiates it; rather, they argue that a thing is beautiful because it participates in Beauty itself. Likewise, in the case of proportion, a Platonist would simply respond that different kinds of mathematical objects can have the same proportion because they participate in the same Form of the proportion.

Nevertheless, the fact that a proportion is instantiated by the mathematical objects of the intermediates can be taken to mark an important difference between proportions and the intermediates. A significant ontological distinction obtains, we should note, between a property itself and its instantiators in Platonism: a property, *F*-ness, exists independently of *F*-things, and *F*-ness is causally or ontologically prior to *F*-things or its embodiments. Thus, it seems that, for Plato, Forms and their instantiators do not belong to the same ontological realm. Then, if a proportion is instantiated by a pair of

[182] 'τὸ συμβέβηκος', which is translated into 'a property' in English, literally means something coming together. The etymology of the term suggests that a property of an individual substance is something whose existence comes together with the existence of an individual substance.

numbers, for instance, the realm of the proportion has to be different from that of the numbers, just as the domain of *F*-ness is different from that of *F*-things. And since *F*-ness is always ontologically prior to its instantiators, proportions should be placed on a higher ontological level than other mathematical objects. Some line of reasoning of this kind may have underlain Aristotle's suspicion in the above passage that Platonists should locate proportions not between intermediates and sensibles, but rather between Forms and intermediates. In other words, Aristotle believes that Platonists should assign a higher ontological status to proportions than to intermediates.

A Platonist might respond that the mathematical objects' instantiation of proportions cannot be seen simply as ontologically of the same kind as sensible objects' instantiation of mathematicals; while a proportion is perfectly embodied in a pair of mathematical quantities, a sensible object never perfectly instantiates a mathematical object, e.g., a pair of numbers will perfectly realize a mathematical ratio, whereas a bronze sphere only imitates a mathematical circle. Aristotle might argue, however, that, according to ASF the motivation for founding a separate domain of scientific objects beyond the sensible world was that sensible objects do not satisfy the conditions for being scientific objects. But since the proportions instantiated by mathematical objects are exactly those dealt with by the theory of proportion, there should be no need to posit ideal proportions beyond their instantiations.

Aristotle also exploits the inseparability of proportions for his argument for the inseparability of mathematical objects. Let us allow that if a proportion does not exist separated from its instantiations, the theory of proportion does not give rise to the problem of multiplication. Platonists, then, might accept that there is no ideal proportion beyond its instantiation. But given that no proportion may be said to exist inseparated from its instantiator, Aristotle argues, there is no reason to hold that other mathematical objects stand apart from sensible objects, either.[183] So, according to Aristotle's argument, Platonists have a problem with the general theory of proportion construed either way: If proportions exist separated from their instantiations, Platonism cannot avoid the problem of multiplication;[184] if they do not, other mathematical objects may also exist unseparated from their instantiations, i.e., from sensible objects.

Nevertheless, it does not seem that either horn of the dilemma is fatal to mathematical Platonism. First, the multiplication problem raised by the general theory of proportion need not be intolerable for Platonists. Since it does not involve an infinite regress in the way that the Third Man argument does, Platonists only have to accept the existence of another kind of mathematical substances in addition to the intermediates. That is, to posit another mathematical kind besides the intermediates threatens no contradiction or inconsistency for mathematical Platonism. Likewise, the second horn of the dilemma is manageable: if the relationship between proportions and mathematical quantities is not the same as that governing the instantiation of a mathematical object by a sensible, it does not follow from that fact that the objects of the general theory of proportion are not separated from numbers or magnitudes that other mathematical objects may not be separate from sensible objects. This was the second question we raised before.

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[183] *Met.*, XIII, 3, 1077b17-23.
[184] *Ibid.*, 2, 1077a9-15.

It is tempting to think that if the objects of mathematical theory on its highest level of generality are not separated from sensibles, it should be the case that other mathematical objects on lower levels of generality should likewise be inseparable from their embodiments. Nevertheless, although one of the features of a Form is universality, it is not clear whether it is a Platonic thesis that a more general concept is more likely to have its Form than a less general one is, e.g., that it is more probable for the Form of animal to exist than for the Form of human to exist;[185] there is no explicit Platonic account of the relationship between Forms. Similarly, we cannot decide whether it is a Platonic thesis that a more general mathematical property is more likely to exist beyond its instantiation than a less general one is.

To evaluate the success of Aristotle's argument from the general theory of proportion, however, we need to consider its aim. If the argument was designed to

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[185] In *Parmenides*, Plato accepts Forms of likeness, unlikeness, unity, plurality, motion and rest (*Parmenides*, 129d-e); whereas he is perplexed over whether Forms of man, fire, and animals, exist; and denies Forms of hair, mud, and dirt (*Ibid*., 130c). One distinguishing feature of the first class of Forms is that they are extremely general; the other classes of Forms could be said to be relatively specific. However, it is not clear whether Plato is taking generality as a criterion for the existence of a Form. For instance, while he accepts Forms of moral concepts such as goodness, beauty, and justice as well (*Ibid*., 130b), this class of Forms could be contrasted with the second group of Forms not on account of degree of generality, but rather because the latter are undignified and worthless. Moreover, it is not certain whether Plato has a consistent criterion by which to decide which concepts have Forms and which do not; in *Republic*, he seems to assume a Form corresponding to every general noun (*Republic*, X, 596a). The *Timaeus* recognizes Forms of the four elements including fire (*Timaeus*, 51c); the *Philebus* (*Philebus*, 15a) accepts Forms of man and ox, etc.

In the later *Dialogues* such as *Parmenides, Sophist* and *Philebus*, Plato attempts to explain the relationship among Forms, laying no especial stress on Forms' independence. Nonetheless, even in the later works, there is no systematic account of the relationship between Forms. Thus, it is far from clear whether there is a hierarchy of Forms or not.

provide a decisive proof against mathematical Platonism, it would be reasonable to consider it a failure. As mentioned earlier, however, *Met*. XIII and XIV can be best characterized as polemic or dialectic; their main purpose is to persuade or refute particular opponents. Viewed in this light, the argument is fairly persuasive and effective. Due to lack of textual evidence, we do not know, either, whether Platonists regarded the problem of multiplication raised by the general theory of proportion as problematic to their view; or whether they invoked generality as a criterion for the existence of a Form. If either or both of these are the case, however, the general theory of proportion furnishes a strong objection against mathematical Platonism. We also considered previously Aristotle's argument that the objects of applied mathematical sciences are not separated from sensible objects. In the text, that argument (*Met*., XIII, 2, 1077a1-9) immediately precedes Aristotle's naïve realistic interpretation of the general theory of proportion (*Ibid*., 1077a9-15). When the objects of mathematical sciences on the highest level of generality, on the one hand, and on the lowest, on the other hand, are not substances in the sense that their existence rests on the existence of something else, it will hardly be likely that only the intermediates, located between the objects of the general theory of mathematics and those of applied mathematics in terms of generality, will enjoy the status of substance in existing separate from sensibles.

Besides, regardless of the validity of the argument, the ontological implications Aristotle tries to draw from the general theory of proportion are clear: mathematical objects do not exist separated from sensible objects. Mathematicals cannot thereby be substances, for only substances exist by themselves, while the other categorical beings depend on them ontologically. Thus it seems difficult to deny that Aristotle supports mathematical naïve realism given that he argues that: (i) mathematical objects exist, (ii) they do not exist separated from sensible objects, (iii) they exist as quantity, a kind of property of substance, and (iv) we can experience them; they are perceived by our senses.[186] No careful reader of Aristotle's mathematical texts can deny this naïve realistic element in his philosophy of mathematics.

3. Problems with Aristotle's Naïve Realism

However, this is only half of the story. It seems indubitable, that is, that Aristotle's theory of mathematics starts out from naïve realism; and indeed he makes every effort to maintain this stance. Naïve realism in mathematics is an attractive position for Aristotle; it is anti-Platonic and suits his scientific realism. Although there are commentators who assimilate Aristotle's view on mathematics to naïve realism, however, it would also be an oversimplification to equate his views of mathematics with that position entirely. Aristotle's mathematical naïve realism raises several problems, which Aristotle himself recognized. Attempting to provide solutions for those problems, he modified and complicated his original position; as a result, his theory came to involve quite different commitements, ones apparently hardly compatible with naïve realism.

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[186] For the naïve empirical realistic interpretation, see Mignucci (1987) p. 187; Barnes (1985) p. 107.

One issue at stake is whether his naïve realism could accommodate such modification and complication without losing the internal consistency of his theory. Commentators have struggled to provide an interpretation capable of unifying two seemingly incompatible aspects of his theory—a supposedly initial naïve realism and later modifications of Aristotle's view. We will examine their interpretations in the next chapter. In the remainder of this, we will consider the difficulties Aristotle has in maintaining mathematical naïve realism, and why that label is not adequate to all of his claims about mathematics.

One of the grounds of Aristotle's concern with mathematical naïve realism is that it blurrs the line of demarcation between mathematics and physics. According to mathematical naïve realism, mathematics is a study of the properties of sensible objects. But if mathematics and physics both study sensible substances, what distinguishes mathematics from a natural science? Both would seem to discuss different aspects of objects in the same domain, namely, the sensible world.[187] Moreover, some objects of mathematics are treated by physics as well. As Aristotle put it:

> …natural bodies contain surfaces and volumes, lines and points, which a mathematician studies…for it would be absurd if the student of nature were supposed to know what the sun or moon is, but none of their properties in its own right, particularly as it is obvious that those who discuss nature also discuss the shape of the moon and sun, and whether the earth and the cosmos are spherical or not.

[187] Another passage which is suggestive of such naïve realism is *Met.*, XIII, 3, 1078a5-9, which treats mathematical properties in the same way as biological properties; both are regarded as the properties of individual substances.

> ...ἐπίπεδα καὶ στερεὰ ἔχει τὰ φυσικὰ σώματα καὶ μήκη καὶ στιγμάς, περὶ ὧν σκοπεῖ ὁ μαθηματικός...εἰ γὰρ τοῦ φυσικοῦ τὸ τί ἐστιν ἥλιος ἢ σελήνη εἰδέναι, τῶν δὲ συμβεβηκότων καθ' αὑτὰ μηδέν, ἄτοπον, ἄλλως τε καὶ ὅτι φαίνονται λέγοντες οἱ περὶ φύσεως καὶ περὶ σχήματος σελήνης καὶ ἡλίου, καὶ δὴ καὶ πότερον σφαιροειδὴς ἡ γῆ καὶ ὁ κόσμος ἢ οὔ.[188]

Why does Aristotle bother himself with that demarcation problem? For Aristotle, mathematics and physics are supposed to be different sciences. According to Arisotle, ideally all true propositions in a science are deducible from a finite number of primitive principles or axioms (ἀρχή), which are indemonstrable. Aristotle argues that the principles of physics cannot be the same as those of mathematics, because while mathematics deals with unchanging things, physics studies things *qua* movable; and a principle of change cannot be present in unchangeable things or things *qua* unchangeable.[189]

Aristotle explains that mathematics is distinguished from the science of nature because "the mathematician also treats of these things, but not *qua* limits of each natural body (περὶ τούτων μὲν οὖν πραγματεύεται καὶ ὁ μαθηματικός, ἀλλ' οὐχ ᾗ φυσικοῦ σώματος πέρας ἕκαστον)."[190] To illustrate this point, he introduces the famous example of "the snub", which is contrasted with "the concave". Aristotle argues

[188] *Phys*., II, 2, 193b23-30.
[189] *Met*., III, 1, 996a20-2, 996a35; XI, 7, 1064a31-33.
[190] *Phys*., II, 2, 193b31-32.

that, while a natural science studies a nose *qua* snub, mathematics studies it *qua* concave. Since 'snub' is always a description of a nose, the study of snubness is a study of a certain kind of nose i.e., of a natural body; but the study of concavity of a nose does not have to be a study of nose. Let us consider the following sentences:

(1) A studies a nose *qua* snub.

(2) B studies a nose *qua* concave.

According to my analysis of the '*qua*' operator, (1) implies that A studies those properties of a nose which are *kath' hauta* properties of snub, whereas (2) means that B studies those properties of a nose which are the *kath' hauta* properties of something concave. The operation of '*qua*' further rests on the suppositions, as articulated in chapter 2, that X is a *kath' hauto* property of Y iff either X is a property of Y and X appears in the definition of Y, or if X is a property of Y and Y appears in the definition of X.[191] Then, since snubness is always a property of a nose, definitions of 'snub' should include 'nose', i.e., snubness is no more than the concavity of a nose; so that the definition of a *kath' hauto* property of snub also should mention 'nose' directly or indirectly.[192] However, since the

[191] *Apo.*, I, 4, 73a34-38; 6, 74b7-10; 22, 83b13-18. For the discussion of Aristotle's criteria for a *kath' hauto* property, see Ch. Two, Sec. 3.

[192] "For of these the formula of the snub mentions the matter of the thing, but that of the concave is stated apart from the matter; for the snubness exists in a nose, so that its formula includes its nose; for the snub is a concave nose (τούτων γὰρ ὁ μὲν τοῦ σιμοῦ λόγος μετὰ τῆς ὕλης λέγεται τῆς τοῦ πράγματος, ὁ δὲ τοῦ κοίλου χωρὶς τῆς ὕλης: ἡ γὰρ σιμότης ἐν ῥινὶ γίγνεται, διὸ καὶ ὁ λόγος αὐτῆς μετὰ ταύτης θεωρεῖται: τὸ σιμὸν γάρ ἐστι

definition of concavity is not phrased in terms of 'noses', we do not have to refer to noses in order to define any of the *kath' hauta* propertie*s* of concavity. Aristotle put it: "the snub is bound up with matter (for the snub is a concave nose), while concavity is independent of perceptible matter (τὸ μὲν σιμὸν συνειλημμένον ἐστὶ μετὰ τῆς ὕλης (ἔστι γὰρ τὸ σιμὸν κοίλη ῥίς), ἡ δὲ κοιλότης ἄνευ ὕλης αἰσθητῆς)."[193]

One might object that, although some *kath' hauta* properties of snub cannot be separated from its perceptible matter, namely, a nose, some of them can be defined without reference to any of the sensible matter that might make up 'snub'. For instance,

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ῥὶς κοίλη) (*Met.*,XI, 7, 1064a23-26)." For Aristotle's discussion of the difference between snub and concave, see also, *Phys.*, II, 2, 193b23-4a12; *Met.*, VI, 1, 1025b30-33; VII, 5, 1030b28-30; 10, 1035a5-8; 1035a26-30. Many commentators have discussed the example of snub and concavity. See, among others, Cleary (1985) p. 29 and Ross (1924) *Vol. I,* p. lxxviii.

[193] *Met.*, VI, 1, 1025b32-34. Mathematics can be distinguished from a natural science also in that, while the former deals with sensible objects *qua* immovable, the latter treats sensible objects but not *qua* immovable. Aristotle says, "For as there are many formulae [about things] merely *qua* moving, apart from each essence of such things and from their properties, but it is not necessary for this reason that there should be either something moving separate from sensibles, or some separate nature in such things, so there will be also formulae and sciences of moving things not *qua* moving but only *qua* bodies, or again only *qua* planes, or only *qua* lines, or *qua* divisibles, or *qua* indivisibles having position, or only *qua* indivisible (ὥσπερ γὰρ καὶ ᾗ κινούμενα μόνον πολλοὶ λόγοι εἰσί, χωρὶς τοῦ τί ἕκαστόν ἐστι τῶν τοιούτων καὶ τῶν συμβεβηκότων αὐτοῖς, καὶ οὐκ ἀνάγκη διὰ ταῦτα ἢ κεχωρισμένον τι εἶναι κινούμενον τῶν αἰσθητῶν ἢ ἐν τούτοις τινὰ φύσιν εἶναι ἀφωρισμένην, οὕτω καὶ ἐπὶ τῶν κινουμένων ἔσονται λόγοι καὶ ἐπιστῆμαι, οὐχ ᾗ κινούμενα δὲ ἀλλ' ᾗ σώματα μόνον, καὶ πάλιν ᾗ ἐπίπεδα μόνον καὶ ᾗ μήκη μόνον, καὶ ᾗ διαιρετὰ καὶ ᾗ ἀδιαίρετα ἔχοντα δὲ θέσιν καὶ ᾗ ἀδιαίρετα μόνον) (*Ibid.*, XIII, 3, 1077b22-30)." The following passages argue much the same point: *Phys.*, II, 2, 193b35; *Met.*, VI, 1, 1026a28-32; 1026a8-13; XI, 7, 1064a23-1064a35; XIII, 3, 1077b23-30. Also see Ross (1924) *Vo., I,* pp. lxxviii and ciii and Hussey (1991), p. 111.

assuming that the definition of snubness is the concavity of a nose, ‘being concave’ is a *kath’ hauto* property of snubness, and its definition does not contain the concept of ‘nose’. Moreover, ‘being concave’ is also a geometrical property. Thus, although the range of mathematical objects is not identical with that of physical objects, we may be able to say that some properties are dealt with both in mathematics and physics. However, the disciplines would continue to exhibit a difference: Physics talks about concavity only as a property of snub; it does not investigate concavity itself. In other words, the fact that the snub is concave remains a theorem of snubness. To study concavity is a geometer’s job; only geometry studies and seeks to specify *kath’ hauta* properties of concavity.

It seems reasonable to suppose that Aristotle deployed abstraction to solve this problem of the demarcation between physics and mathematics while maintaining his mathematical naïve realism. Certainly, Aristotle had grounds for wanting mathematics and physics to be sciences of different (sets of) objects. Even so, the possible failure of the delimitation of the sciences does not threaten mathematical naïve realism; nor does it harm his anti-Platonism or scientific realism. Even if the boundary between the two sciences collapses, both anti-Platonism and scientific realism could be still preserved. Another problem, though, poses a more severe challenge to Aristotle’s mathematical naïve realism.

I have argued that Aristotle employs abstraction specifically in *Met*. XIII and XIV to criticize AFS by way of refuting mathematical Platonism. Nevertheless, it is not easy to explain how mathematical objects or mathematical properties can be obtained from sensible particulars by abstraction.[194] Notice that A can study B *qua* C only if B is C; e.g., we can study Socrates *qua* man only when he is a man. That is to say, we can obtain a certain *kath' hauto* property of C, say, M, from B by abstraction only when it is true that B is M. This seems to be a trivial point. It is sufficient, however, to complicate Aristotle's philosophy of mathematics. Suppose that A studies B *qua* C, and that A is mathematics, B a sensible particular, and C triangle. Then, there will be no sensible particular which is substitutable for B; simply because there is no perfect instantiation of a triangle in sensible objects. This is the so-called precision problem, the central issue of interpretation of Aristotle's philosophy of mathematics.

Naïve realistic interpreters might raise a couple of objections: First of all, Aristotle's theory of abstraction is based on different ontological assumptions from Plato's. While Plato assumes that a sensible particular cannot perfectly realize a Form, Aristotle's universals are always in sensible particulars. Since, like other scientific objects, mathematical objects are universals, Aristotle would believe that, supposing abstraction can elicit the predicate manhood from Socrates, the same method can elicit triangularity from a bronze isosceles triangle. Put another way, if we cannot abstract triangularity from a bronze isosceles, we cannot abstract manhood from Socrates, either; but Aristotle admits that the latter is possible.

[194] Ross says, "the 'spheres' and 'circles' of common life are not spheres and circles at all, and it is not of them that geometrical proposition are true. Such propositions are true of the perfect geometrical figures which thought recognizes as existing in space though their boundaries do not coincide with those of any sensible figure." See Ross (1924) I, p. lvi. Muller agrees with this point. See Mueller (1970) p. 158.

Nevertheless, there seems to be an obvious difference between the instantiation of a mathematical object and that of an object of other sciences. Let us call A a perfect instantiation of B just in the case that a property of A satisfies the definition of B. Then, it is difficult to deny that there is a perfect instance of a mammal in the sensible world, for instance; but it seems to be hardly possible to find a perfect instantiation of a triangle. Thus, it is fair to say that some mathematical objects cannot be obtained from sensible particulars by abstraction, whereas this problem does not affect other scientific objects.

However, it might be also objected that it is one thing to hold that there can be no perfect instantiation of a mathematical object, but quite another to assign this view to Aristotle. If Aristotle believes that mathematical objects can be realized by sensible particulars just in the manner of the realization of other scientific objects, the precision problem does not occur at least within his metaphysics.

This objection involves two different issues: One is whether Aristotle admits that there exist perfect instantiations of mathematical objects in the sensible world; the other is whether his admission of the existence of such perfect mathematical instantiations exempts his philosophy of mathematics from the precision problem. Let us consider the second issue first.

Suppose that Aristotle's metaphysics assumes that there are perfect physical instances of mathematical objects. Does this assumption allow him to avoid the precision problem? It seems not. Even if sensible substances perfectly realize some geometrical properties, there are many geometrical figures which, it seems, cannot be found in the sensible world; for example, geometry can construct a regular polygon with 100 sides and even discover theorems of it, but it is highly improbable that there exists a physical instance of that regular polygon.[195] If the shape has no physical instantiation in the sensible world, we cannot obtain it from a sensible object by abstraction.[196] Even in the case of circles and straight lines, of which Aristotle claims there are perfect physical instantiations, we are faced with the same problem. Geometry can study any size of circle. But it is impossible that physical objects could exemplify absolutely every size of circle. We can acquire circularity from the trace made by the movement of a heavenly body, but abstraction (in the sense of the progressive elimination of irrelevant properties) cannot help us acquire a circle of a radius of 3 cm from that particular trace. The same argument can be applied to straight lines as well. Thus, it seems that, insofar as Aristotle does not believe in the existence of a perfect physical instantiation of each possible mathematical object, he cannot avoid the precision problem, regardless of whether some mathematical forms are perfectly instantiated in the sensible world. Commentators have provided various divergent solutions for this precision problem. We will examine each of their principal positions in the next chapter.

[195] Mignucci (1987) pp. 187-188.

[196] Beside the precision problem, another objection to Aristotle's idea of the obtaining of mathematical objects from particulars by abstraction is, I think, perhaps more fundamental. If a geometer thinks of a geometrical figure without having any physical instantiation, that geometrical figure cannot be said to be gained from a physical object by abstraction. Thus, it can be said that Aristotelian abstraction has nothing to do with geometry. Hussey further argues that for mathematicians to study mathematics, they do not need to have any experience of the actual world because the structure of mathematical objects is totally determined by a handful of postulates (See Hussey (1992) p. 129). It is notable that Aristotle also argues that young people can be good mathematicians because mathematics requires no experience of the world (*NE.*, VI, 8, 1142a11-21). This is another aspect of Aristotle's view of mathematics to be treated later in this chapter.

4. The Problem of the Internal Consistency of Aristotle's Theory of Mathematics

Let us now return to the first issue. Many scholars have struggled with the question whether Aristotle admits the existence of the perfect physical instantiations of some mathematical objects, believing that if he does, his philosophy of mathematics is immune from the precision problem. But we saw that that is not the case. Rather, to head off the precision problem satisfactorily, Aristotle has to show that every possible mathematical object can be obtained by abstraction from sensible objects. Thus, the question about the existence of some perfect physical instances of particular kinds of mathematicals such as circles, straight lines, triangles, etc., is in no way decisive for the precision problem after all.

Nevertheless, the question is still important. Besides the precision problem, it raises the problem of the internal consistency of Aristotle's philosophy of mathematics, i.e., whether Aristotle has a coherent theory of mathematics or holds more than one conflicting views. On the one hand, there is *prima facie* evidence for Aristotle's belief in the existence of perfect instantiations of some mathematical objects in the sensible world. We already saw Aristotle's argument that the objects of applied mathematical sciences such as astronomy, optics, and harmonics are something acquired from sensible substances by abstraction. For example, the movement of heavenly bodies draws a perfect circle;[197] and the shape of the heaven is of necessity spherical.[198] Aristotle also suggests in some passages that some movements are straight.[199]

> But every motion in place, which we call locomotion, is either straight or circular or a combination of these; for they are only two simple motions. And the reason is that these straight and circular [lines] are the only simple magnitudes. By 'circular [motion]' I mean [motion] around center, whereas 'straight [motion]' means the upward and downward [motion]. While I mean by 'upward' motion away from the centre, 'downward' motion toward it.
>
> Πᾶσα δὲ κίνησις ὅση κατὰ τόπον, ἣν καλοῦμεν φοράν, ἢ εὐθεῖα ἢ κύκλῳ ἢ ἐκ τούτων μικτή: ἁπλαῖ γὰρ αὗται δύο μόναι. Αἴτιον δ᾽ ὅτι καὶ τὰ μεγέθη ταῦτα ἁπλᾶ μόνον, ἥ τ᾽ εὐθεῖα καὶ ἡ περιφερής. Κύκλῳ μὲν οὖν ἐστιν

[197] *DC*.,II, 4, 287a11-22.

[198] *Ibid.*, 286b10-287b21.

[199] Another passage where Aristotle seems to think that there are perfectly straight physical objects is found at *DA*., I, 1, 403a10-16. He says, "Thus, if there is any of functions or affections peculiar to the soul, it will be possible for the soul to be separated [from the body]; if there is none peculiar to it, it will not be separable, just as many [properties] belong to the straight *qua* straight, e.g., *touching a bronze sphere at a point*, but if something straight is separated, it will not touch in this way; for it is inseparable, since it is always with some body (εἰ μὲν οὖν ἔστι τι τῶν τῆς ψυχῆς ἔργων ἢ παθημάτων ἴδιον, ἐνδέχοιτ᾽ ἂν αὐτὴν χωρίζεσθαι: εἰ δὲ μηθέν ἐστιν ἴδιον αὐτῆς, οὐκ ἂν εἴη χωριστή, ἀλλὰ καθάπερ τῷ εὐθεῖ, ᾗ εὐθύ, πολλὰ συμβαίνει, οἷον *ἅπτεσθαι τῆς [χαλκῆς] σφαίρας κατὰ στιγμήν*, οὐ μέντοι γ᾽ ἅψεται οὕτως χωρισθέν τι εὐθύ: ἀχώριστον γάρ, εἴπερ ἀεὶ μετὰ σώματός τινος ἐστιν)."

Commentators disagree on the interpretation of this passage. While Lear takes this passage as evidence for Aristotle's belief in the perfect physical instance of mathematical straight line (Lear (1982) pp. 180-181), Ross excises 'τῆς χαλκῆς' from the sentence 'ἅπτεσθαι τῆς χαλκῆς σφαίρας κατὰ στιγμήν,' and reads it as meaning that straightness abstracted from the straight thing touches a (geometrical) sphere at a single point (See Ross (1961) p. 168). Many commentators, however, think that τῆς χαλκῆς is essential for a correct reading. See Mansion (1978) pp. 1-20; Mckay (1979) pp. 86-91; Hamlyn (1993) pp. 78-79.

> ἡ περὶ τὸ μέσον, εὐθεῖα δ᾽ ἡ ἄνω καὶ κάτω. Λέγω δ᾽ ἄνω μὲν τὴν ἀπὸ τοῦ μέσου, κάτω δὲ τὴν ἐπὶ τὸ μέσον.[200]

However, he also suggests in several places that sensible substances do not fulfill the conditions that mathematical objects should meet. For instance, he says:

> But on the other hand astronomy cannot be concerned with sensible magnitude nor with this heaven. For such sensible lines are never like those of which the geometer speaks (for none of sensible things is thus straight or round: for a [physical] circle touches a straight edge not at a point but as Protagoras said in his refutation of the geometers) nor are the motions and spiral orbits of the heaven like those which astronomy investigates nor have [geometrical] points the same nature as stars.
>
> ἀλλὰ μὴν οὐδὲ τῶν αἰσθητῶν ἂν εἴη μεγεθῶν οὐδὲ περὶ τὸν οὐρανὸν ἡ ἀστρολογία τόνδε. οὔτε γὰρ αἱ αἰσθηταὶ γραμμαὶ τοιαῦταί εἰσιν οἵας λέγει ὁ γεωμέτρης (οὐθὲν γὰρ εὐθὺ τῶν αἰσθητῶν οὕτως οὐδὲ στρογγύλον: ἅπτεται γὰρ τοῦ κανόνος οὐ κατὰ στιγμὴν ὁ κύκλος ἀλλ᾽ ὥσπερ Πρωταγόρας ἔλεγεν ἐλέγχων τοὺς γεωμέτρας), οὔθ᾽ αἱ κινήσεις καὶ ἕλικες τοῦ οὐρανοῦ ὅμοιαι περὶ ὧν ἡ ἀστρολογία ποιεῖται τοὺς λόγους, οὔτε τὰ σημεῖα τοῖς ἄστροις τὴν αὐτὴν ἔχει φύσιν.[201]

This passage not only denies perfect instances of mathematical properties such as straightness and roundness, but also claims that the objects of astronomy are not physical objects. This implies exactly the opposite of those passages gathered to support the naïve

[200] *DC*., I, 2, 268b17-22.
[201] *Met*., III, 2, 997b34-998a6.

realistic interpretation. So, some scholars take this passage as textual evidence that Aristotle thinks that mathematics does not directly describe physical objects.[202]

Some commentators attempt to preserve the consistency of Aristotle's view on mathematics by denying that, for Aristotle, physical objects do not perfectly instantiate mathematical properties. For instance, Lear argues that the passage quoted above "need not to be construed as supporting the thesis that physical objects fall short of truly possessing geometrical properties"; since the context shows that the passage presents a view of Platonists' rather than Aristotle's own opinion.[203] At *Met*., III,2, 997a34-35, after asking the question whether there are other kinds of substances besides sensible substances, Aristotle criticizes those who assert the existence both of the Forms and of the intermediates; this criticism goes on as far as 998a20. Thus, the passage above (997b34-998a6) should be seen as part of the criticism; it shows the absurdity of assuming the existence of other substances than sensible ones, i.e., under this supposition, astronomy would be led to study something other than the physical objects such as heavenly bodies and their movements. This reading is confirmed by the following passage, which immediately precedes the passage above.

> Again, if someone is going to posit the intermediates apart from and between the Forms and the sensible, he shall have many difficulties. For it is evident that similarly there will be lines apart from these Forms of lines and sensible lines and so with each of the other kinds. Thus, since astronomy is one of these [mathematical sciences] there will also be some

[202] See Annas (1976) p. 29. For interpretations of this passage, see also Cleary (1945), pp, 424-94; Madigan (1999), pp. 58-59.
[203] Lear (1982) p. 176.

heaven apart from the sensible heaven, and a sun and a moon and similarly the other heavenly bodies [apart from the sensible one] along with the [separate] heaven. Yet how should we believe these things? For it is not reasonable that [these separate things] are immovable, but it is altogether impossible that they are moving. And similarly with those things which optics and mathematical harmonics deal with. For it is impossible for these things to exist apart from the sensible things, for the same reasons.

ἔτι δὲ εἴ τις παρὰ τὰ εἴδη καὶ τὰ αἰσθητὰ τὰ μεταξὺ θήσεται, πολλὰς ἀπορίας ἕξει: δῆλον γὰρ ὡς ὁμοίως γραμμαί τε παρά τ᾽ αὐτὰς καὶ τὰς αἰσθητὰς ἔσονται καὶ ἕκαστον τῶν ἄλλων γενῶν: ὥστ᾽ ἐπείπερ ἡ ἀστρολογία μία τούτων ἐστίν, ἔσται τις καὶ οὐρανὸς παρὰ τὸν αἰσθητὸν οὐρανὸν καὶ ἥλιός τε καὶ σελήνη καὶ τἆλλα ὁμοίως τὰ κατὰ τὸν οὐρανόν. καίτοι πῶς δεῖ πιστεῦσαι τούτοις; οὐδὲ γὰρ ἀκίνητον εὔλογον εἶναι, κινούμενον δὲ καὶ παντελῶς ἀδύνατον: ὁμοίως δὲ καὶ περὶ ὧν ἡ ὀπτικὴ πραγματεύεται καὶ ἡ ἐν τοῖς μαθήμασιν ἁρμονική: καὶ γὰρ ταῦτα ἀδύνατον εἶναι παρὰ τὰ αἰσθητὰ διὰ τὰς αὐτὰς αἰτίας.[204]

Clearly, this passage states that it is not reasonable to suppose that there exist objects of astronomy other than sensible heavenly bodies. Since this passage is an integral part of the whole criticism of 997a34-998a20, it would be appropriate to read the previous passage in such a way as to make it consistent with the latter.

However, there are other passages which sit much less easily with the naïve realistic interpretation. For example, Aristotle says:

[204] *Met.*, III, 2, 997b12-23.

> …what sort of thing is the mathematician supposed to deal with? Clearly not with things in this world; for none of these is the sort of thing which the mathematical sciences investigate.
>
> …περὶ ποῖα θετέον πραγματεύεσθαι τὸν μαθηματικόν; οὐ γὰρ δὴ περὶ τὰ δεῦρο. τούτων γὰρ οὐθέν ἐστιν οἷον αἱ μαθηματικαὶ ζητοῦσι τῶν ἐπιστημῶν.[205]

Lear argues that this passage once more does not present Aristotle's own view; rather, it is the Academic Platonist's response to the objection that the postulation of the intermediates involves a range of absurdities.[206] The context in which the passage appears, though, offers no grounds for supposing that the speaker of these views is anyone other than Aristotle himself. Having criticized a Platonic doctrine that mathematical objects are intermediates between Forms and sensible objects in the immediately preceding passage (1059a5-10), Aristotle, next, asks himself what kind of thing mathematical objects are supposed to be and answers that they are not things of this world.

Any need to engage closely with Lear's interpretation of this passage is lessened by the existence, in fact, of a number of similar passages in which Aristotle makes comparable assertions in unambiguously his own voice, such as the following:[207]

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[205] *Ibid*., XI, 1, 1059b10-12.

[206] Lear (1982) p. 178-179.

[207] Similarly, he says, "So, too is it with geometry; even if things with which it is concerned happen to be sensible objects, though not *qua* sensibles, mathematical sciences will not be concerned with sensible objects—nor, on the other hand, with other things separated from these [sensibles] (οὕτω καὶ τὴν γεωμετρίαν: οὐκ εἰ συμβέβηκεν αἰσθητὰ εἶναι ὧν ἐστί, μὴ ἔστι δὲ ᾗ αἰσθητά, οὐ τῶν αἰσθητῶν ἔσονται αἱ μαθηματικαὶ ἐπιστῆμαι, οὐ

> It has, then, been sufficiently shown that the objects of mathematics are not substances more than bodies are, and that they are not prior to sensibles in being, but only in formula, and that they cannot in any way exist separately. But since *they could not exist in sensibles* either, it is evident that they either do not exist at all or exist in some way, and for this reason they do not exist absolutely for 'exist' is said in many ways.
>
> Ὅτι μὲν οὖν οὔτε οὐσίαι μᾶλλον τῶν σωμάτων εἰσὶν οὔτε πρότερα τῷ εἶναι τῶν αἰσθητῶν ἀλλὰ τῷ λόγῳ μόνον, οὔτε κεχωρισμένα που εἶναι δυνατόν, εἴρηται ἱκανῶς: ἐπεὶ δ᾽ οὐδ᾽ ἐν τοῖς αἰσθητοῖς ἐνεδέχετο αὐτὰ εἶναι, φανερὸν ὅτι ἢ ὅλως οὐκ ἔστιν ἢ τρόπον τινὰ ἔστι καὶ διὰ τοῦτο οὐχ ἁπλῶς ἔστιν: πολλαχῶς γὰρ τὸ εἶναι λέγομεν.[208]

The whole paragraph constitutes an argument, which can be formulated as follows:

> (1) The objects of mathematics cannot exist separate from sensibles; the former are not ontologically prior to the latter;
>
> (2) They are not in sensibles.
>
> (3) Therefore, they do not exist absolutely.

Putting it this way, (2) is one of premises from which the conclusion (3) is drawn. So, if we accept (3) as Aristotle's own thesis, we should admit that (2) as also a claim made by Aristotle's philosophy.

μέντοι οὐσὲ παρὰ ταῦτα ἄλλων κεχωρισμένων) (*Met*., XIII, 3, 1078a2-5)." cf. *Ibid*.,III, 5, 1002a15-18; 20-29.

[208] *Ibid*., XIII, 2, 1077b12-16.

Besides the question whether (2) is Aristotle's assertion or not, it is questionable how (3) derives from (1) and (2). To deal with this, we need to first clarify what Aristotle means by saying that: (i) A does not exist separate from B, and (ii) A is in B. In Aristotle's ontology, whatever exists must belong to one of the ten categories; of them, only substance can exist by itself; the other categories of beings exist only along with substances, never separately from them. In other words, beings other than substance are something that belong to a substance, and that a substance will bear as its properties. It seems to be least questionable that the sensible objects of the above passage are sensible substances; Aristotle's philosophy of mathematics begins with asking what other substances exist *besides* sensible substances. Thus, his assertion that mathematical objects cannot exist separately from sensibles can plausibly be taken to mean either: (iii) such objects are not substances, or (iv) such objects exist as the properties of sensible substances.

However, (iv) is excluded by (2). According to Aristotle, A is said to be in (ἐν) B many ways.[209] The most relevant sense is found in *Met.*, V, 23, where Aristotle notes that "'to be in something' is used in similar, and corresponding, ways to 'to have' (τὸ ἔν τινι δὲ εἶναι ὁμοτρόπως λέγεται καὶ ἑπομένως τῷ ἔχειν),"[210] i.e., A is in B just in

[209] Aristotle distinguishes eight different uses of 'being in' (*Phys*., IV, 3, 210a14-b7). He argues that its primary sense marks a spatial relationship of containment, e.g., something is in a vessel. But there is no account of how the other senses are derived from (or otherwise related to) this primary sense. The English 'in' does not exactly corresponds to the Greek, 'ἐν.' Aristotle explains the uses of 'in' also in *Met.*, V, 23, 1024a23-25.
[210] *Ibid*., 1023a24-25.

the case that B has A. He enumerates four uses of ‘to have’; in one, A is said to have B when B belongs to A and A is a kind of ‘recipient’ of B, e.g., bronze has the form of the statue, or a body has a disease.[211] In this sense, if (2) is true, then it will also be true that no sensible substance has a mathematical object in itself, which means that mathematical objects do not belong to sensible substances. Since, then, mathematical objects are neither substances nor belong to sensible substances, (2) implies, after all, that the objects of mathematics cannot be some properties of sensible objects. This is a problem; the implication hardly seems compatible with the mathematical naïve realism which Aristotle elsewhere holds.

Furthermore, several passages in Aristotle seem to argue against locating mathematical objects in sensible bodies. The most informative is found in his criticism of the Platonic position that the intermediates exist in physical objects:[212]

> It has been already observed in the *Discussion of Problems* that it is impossible [for mathematical objects] to exist in sensible things and at the same time that the theory in question is a fabrication, seeing that (a) it is impossible for two solids to be in the same place, and that (b) according to the same theory all the other powers and characteristics are also to exist in sensible things and none of them exist separately. This is what we have seen so far. But, in addition to these, it is obvious that (c) it is impossible for any body whatever to be divided; for it is to be divided at a plane, and the plane at a line, and the line at a point, so that if we cannot divide the point, neither can line, and if we cannot divide the line, neither can others. What difference then does it make whether sensible things are such kind of things, or, they have such things in themselves, without being such kind

[211] *Ibid.*, 1023a11-13.

[212] Annas calls this position ‘partial Platonism’. This is one of the ‘Platonic’ positions which diverged from Plato’s original theory. Another argument against the idea that the intermediates are in physical objects is at *Met.*, III, 2, 998a6-20.

of things? The result will be the same; for when the sensible things are divided they will be divided, or else the sensible things cannot be divided.

῞Οτι μὲν τοίνυν ἔν γε τοῖς αἰσθητοῖς ἀδύνατον εἶναι καὶ ἅμα πλασματίας ὁ λόγος, εἴρηται μὲν καὶ ἐν τοῖς διαπορήμασιν ὅτι δύο ἅμα στερεὰ εἶναι ἀδύνατον, ἔτι δὲ καὶ ὅτι τοῦ αὐτοῦ λόγου καὶ τὰς ἄλλας δυνάμεις καὶ φύσεις ἐν τοῖς αἰσθητοῖς εἶναι καὶ μηδεμίαν κεχωρισμένην: ταῦτα μὲν οὖν εἴρηται πρότερον, ἀλλὰ πρὸς τούτοις φανερὸν ὅτι ἀδύνατον διαιρεθῆναι ὁτιοῦν σῶμα: κατ᾽ ἐπίπεδον γὰρ διαιρεθήσεται, καὶ τοῦτο κατὰ γραμμὴν καὶ αὕτη κατὰ στιγμήν, ὥστ᾽ εἰ τὴν στιγμὴν διελεῖν ἀδύνατον, καὶ τὴν γραμμήν, εἰ δὲ ταύτην, καὶ τἆλλα. τί οὖν διαφέρει ἢ ταύτας εἶναι τοιαύτας φύσεις, ἢ αὐτὰς μὲν μή, εἶναι δ᾽ ἐν αὐταῖς τοιαύτας φύσεις; τὸ αὐτὸ γὰρ συμβήσεται: διαιρουμένων γὰρ τῶν αἰσθητῶν διαιρεθήσονται, ἢ οὐδὲ αἱ αἰσθηταί.[213]

Aristotle here offers three objections to the idea that mathematical objects are in sensible things. It is generally agreed that (a) and (b) resume the two arguments of *Met.,* B, 2*,* where Aristotle asserts:

And there are some who say that these so-called intermediates between the Forms and the perceptible things exist, not apart from the sensible things, but in them; it would require a long discussion to go through all the impossible results of this view, but it is enough to consider the following: (e) It is not reasonable that this is the case only for these [intermediates]; it is clear that Forms also can be in the sensible things (for both of these are explained by the same account). Further, (f) it is necessary that there are two solids in the same place, and (g) that the intermediates are not immovable, while they are in moving things, i.e., sensible things.

εἰσὶ δέ τινες οἵ φασιν εἶναι μὲν τὰ μεταξὺ ταῦτα λεγόμενα

[213] *Ibid*., XIII, 2, 1076a38-b11.

> τῶν τε εἰδῶν καὶ τῶν αἰσθητῶν, οὐ μὴν χωρίς γε τῶν αἰσθητῶν ἀλλ᾽ ἐν τούτοις: οἷς τὰ συμβαίνοντα ἀδύνατα πάντα μὲν πλείονος λόγου διελθεῖν, ἱκανὸν δὲ καὶ τὰ τοιαῦτα θεωρῆσαι. οὔτε γὰρ ἐπὶ τούτων εὔλογον ἔχειν οὕτω μόνον, ἀλλὰ δῆλον ὅτι καὶ τὰ εἴδη ἐνδέχοιτ᾽ ἂν ἐν τοῖς αἰσθητοῖς εἶναι (τοῦ γὰρ αὐτοῦ λόγου ἀμφότερα ταῦτά ἐστιν), ἔτι δὲ δύο στερεὰ ἐν τῷ αὐτῷ ἀναγκαῖον εἶναι τόπῳ, καὶ μὴ εἶναι ἀκίνητα ἐν κινουμένοις γε ὄντα τοῖς αἰσθητοῖς.[214]

We find in the first passage of *Met*., XIII, 2 that (a) and (b) correspond to (f) and (e) in the second passage of *Met*., III, 2, respectively, and (g) is omitted while (c) is newly added. Further, the Forms mentioned in (e) are replaced by the the powers and characteristics in (b).

The arguments developed in the second passage are apparently directed against a particular Platonic position (which Annas terms ‘partial Platonism’),[215] that the intermediates are *in* sensible objects; (e), (f), and (g) show what absurdities follows from the idea that the intermediates are in physical objects. Thus, if the assumption that (a) and (b) correspond to (f) and (e) in *Met, B,* is correct, the theory under fire from Aristotle in this passage would again be partial Platonism. Thus his thesis that mathematical objects are not in sensibles would not be incompatible with his naïve realistic claims, since Aristotle’s mathematical objects are not the intermediates.

[214] *Met.*, III, 2, 998a7-998a15.

[215] See Chapter Three, §4, n. 212.

However, argument (c), which is newly added, does not restrict itself to the intermediates among possible mathematical objects. According to (c), cutting a body involves cutting a plane, cutting a plane involves cutting a line, and cutting a line involve cutting a point; but a point cannot be divided. Thus, if a mathematical body is actually in a physical body and a physical body can be divided, we should accept that a mathematical point can be divided, which is absurd. Since the validity of the argument relies on the assumption that a point is indivisible, whether or not the mathematical objects purportedly in some physical bodies are the intermediates or not is irrelevant. Even if these objects are Forms, the same absurdity follows unless the Form of a point admits of subdivision. In this regard, Annas rightly argues that the argument is not limited to the intermediates but applies to any kind of ideal mathematical object.[216]

Nevertheless, as White argues, the scope of (c) seems to be much broader than Annas takes it to be, whatever Annas means by 'ideal'. Since the geometrical definition of 'point' excludes its divisibility, if (c) is valid, geometrical objects cannot be in physical things; it makes no difference whether the mathematical objects contained in sensibles are ideal or not.[217] Or, at least, (c) implies that geometrical objects cannot be in sensible objects as their actual extended parts.[218]

Since Aristotle here holds that mathematical objects are not in sensible objects, his mathematical naïve realism has to make space for a view apparently contradicted by

[216] Annas (1976) pp. 138-139.
[217] White (1993) pp. 170-172.
[218] At this point, it is possible to broach the possibility that mathematical objects exist in sensible objects as another mode of being, i.e., that they exist as potentiality in sensible objects.

his fundamental position. Thus, if Aristotle has any consistent theory of mathematics, he should have given up either his naïve realism or the view that mathematics is not concerned with sensible objects. It is my hypothesis that Aristotle gave up his original naïve realism at some point and developed another theory. There are two reasons for positing this hypothesis: First, naïve realism cannot evade the philosophical problems we have considered; and as the text indicates, Aristotle was cognizant of these problems. Second, it is possible to reconstruct a consistent theory of mathematics out of his claims which excludes naïve realism. We will see this in the next chapter.

But if the objects of mathematics exist neither as substances nor as properties of substances, then how can they exist at all? Anything existent should belong to one of the categories. Rather than answering this question directly, Aristotle says:

> If the objects of mathematics exist, then they must exist either in sensible objects, as some say, or separate from sensible objects (and some say in this way, too), or if they exist in neither of these ways, either they do not exist, or they exist in some other way. Thus, the subject of our discussion will be not on whether they exist but on the way [in which they exist].
>
> ἀνάγκη δ', εἴπερ ἔστι τὰ μαθηματικά, ἢ ἐν τοῖς αἰσθητοῖς εἶναι αὐτὰ καθάπερ λέγουσί τινες, ἢ κεχωρισμένα τῶν αἰσθητῶν (λέγουσι δὲ καὶ οὕτω τινές): ἢ εἰ μηδετέρως, ἢ οὐκ εἰσὶν ἢ ἄλλον τρόπον εἰσίν: ὥσθ' ἡ ἀμφισβήτησις ἡμῖν ἔσται οὐ περὶ τοῦ εἶναι ἀλλὰ περὶ τοῦ τρόπου.[219]

[219] *Met.*, XIII, 1076a32-37.

He enumerates four possibilities: (i) mathematical objects do not exist, (ii) they exist in sensible objects, i.e., they are properties of sensible substances, (iii) they exist separated from sensible objects, i.e., they are another kind of substance, and (iv) they exist in some other way (ἄλλον τρόπον εἰσίν). He eliminates the first possibility without further argument as incompatible with his scientific realism; insofar as mathematics is a legitimate science, it must have existent objects. But since, as we saw, Aristotle denies (ii) and (iii) in the previous passage (1077b12-16), only (iv) remains.

What kind of beings, then, can mathematical objects be if they are neither substances nor properties of substances? We saw that Aristotle's initial answer to this question is that they do not exist absolutely.[220] But what does he mean by saying that something exists 'not absolutely'? Aristotle elsewhere clarifies that 'to exist absolutely (εἶναι ἁπλῶς)' means 'to exist in respect of substance (εἶναι κατ' οὐσιαν).'[221] This does not get us anywhere; we have already argued that mathematical objects do not exist as substances. But the following passage gives us some hint.

> That is why the geometers speak correctly: they talk about existing things and they do exist, for what exists does in two ways, either in actuality or as material.
>
> διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν: διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ' ὑλικῶς.[222]

[220] *Ibid.*, XIII, 2, 1077b12-16.
[221] *Ibid.*, IX, 8, 1050b16.
[222] *Ibid.*, XIII, 3, 1078a28-31.

Aristotle’s answer is that mathematical objects exist as matter or in a matter-like way (ὑλικῶς). But what he means by ‘matter-like’ is confusing.

First, it is far from clear why he uses the term ‘matter-like’ instead of ‘in potentiality.’ It is very tempting to read ‘matter-like’ as ‘in potentiality’ because: (i) the contrast with ‘actuality’ would make us expect ‘in potentiality’ in the place of ‘matter-like’; (ii) Aristotle frequently uses the terms ‘matter’ and ‘potentiality’ coextensively, [223] and. (iii) he suggests the idea that mathematical objects exist as matter as an answer to the question what kind of being they are if neither substances nor some other category of beings in his ontology, where potentiality figures as another mode of being.[224] A relevant passage runs:

> …‘being’ and ‘non-being’ are said on the one hand in accordance with the figures of the categories and on the other in accordance with the potentiality or actuality of these or of their opposites…
>
> …τὸ ὂν λέγεται καὶ τὸ μὴ ὂν τὸ μὲν κατὰ τὰ σχήματα τῶν κατηγοριῶν, τὸ δὲ κατὰ δύναμιν ἢ ἐνέργειαν τούτων ἢ τἀναντία…[225]

[223] *Phys*, III, 6, 206a14; 206b13-16; *DA*., II, 1, 412a6-11; *Met*., III, 5, 1002a20-25; VIII, 2, 1042b8-11; 1043a14-17; IX, 6, 1045a30-35; 7, 1049a19-24; 8, 1050a15-16; XII, 4, 1070b10-13; XIII, 3, 1078a28-31; XIV, 1088b1-2. For a discussion of Aristotle’s assimilation of ‘matter’ to ‘potentiality’, see Chapter Four, §3, 3.1 and 3.2.

[224] For discussion of potentiality as another mode of being, see Chapter Four, §3, 3.3.

[225] *Met*., IX, 10, 1051a33-1051b1.

Nevertheless, it is still quite possible that Aristotle intentionally chooses a different term in order to indicate something other than potentiality; and even if by ‘matter’ he means ‘potentiality’, it is not clear, either, in what sense mathematical objects exist in potentiality, since the term ‘potentiality’ also has more than one sense in Aristotle.[226] Secondly, if mathematics studies something obtained from a sensible object by abstraction, then this should be the form of the sensible object rather than its matter, e.g., when a geometer studies a bronze sphere, he attends to its form, namely, the sphericity of the bronze, not its bronze matter. Thirdly, unless mathematics is a study of mere *possibilia,* the triangles considered by geometry, for example, must be actually existent triangles. If we are to have knowledge of them, it is consonant with Aristotle’s realism vis-à-vis the sciences that they must exist. But if mathematical objects do not exist in actuality in sensible objects, where do they then exist in actuality?

These difficulties arise due to an ambiguity attaching to the concept, ‘matter’. This ambiguity, however, provides room for various interpretations. As we will see in the next chapter, this short passage has prompted especially voluminous commentary in relation to interpreters’ broad sense of Aristotle’s philosophy of mathematics; on the one hand, commentators have sought to make the passage congruent with Aristotle’s other assertions, while on the other, they have exploited it more tendentiously as the foundation of their own lines of exegesis.

[226] For different senses of ‘potentiality,’ see Chapter Four, §3, 3.2.

Although it is not easy to work out what it means to say that mathematicals exist in a matter-like way or as matter, it is obvious that this passage cannot easily be made to conform with the naïve realistic interpretation. If geometry is the study of the properties of sensible objects, it must be the study of their shapes; if the dichotomy of form and matter is applicable in this case, then objects' shapes should be identified with their forms, not their matter. Thus, Aristotle's claim that mathematicals exist as matter could be seen as textual evidence that Aristotle did not think mathematics is the study of the properties of sensible things.

5. Idealism in Aristotle's View of Mathematics

What is, then, his alternative view to his naïve realism? Aristotle elsewhere provides us with passages which imply a kind of mathematical idealism. Among these, the following is especially informative.

> Geometrical constructions are also discovered in actuality because they (mathematicians) discover [them] by dividing [the given figures]. If they had been already divided, they would have been obvious; but as it is they are in there potentially. Why is the triangle two right angles? It is because the angles around one point are equal to two right angles. Thus, if the line parallel to the side had been drawn up, the reason would have been clear immediately on seeing [it]. Why is there universally a right angle in the semi-circle? Because if three lines are equal, the two which are the base and the one set upright from the centre, it is clear on seeing it to one who knows that. Thus, it is evident that the things in potentiality are discovered by being brought into actuality; the reason is that thinking is the actuality, so that the potentiality is from actuality, and because of this

> they know by constructing, for the individual actuality is posterior in generation.
>
> εὑρίσκεται δὲ καὶ τὰ διαγράμματα ἐνεργείᾳ: διαιροῦντες γὰρ εὑρίσκουσιν. εἰ δ᾽ ἦν διῃρημένα, φανερὰ ἂν ἦν: νῦν δ᾽ ἐνυπάρχει δυνάμει. διὰ τί δύο ὀρθαὶ τὸ τρίγωνον; ὅτι αἱ περὶ μίαν στιγμὴν γωνίαι ἴσαι δύο ὀρθαῖς. εἰ οὖν ἀνῆκτο ἡ παρὰ τὴν πλευράν, ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί. ἐν ἡμικυκλίῳ ὀρθὴ καθόλου διὰ τί; ἐὰν ἴσαι τρεῖς, ἥ τε βάσις δύο καὶ ἡ ἐκ μέσου ἐπισταθεῖσα ὀρθή, ἰδόντι δῆλον τῷ ἐκεῖνο εἰδότι. ὥστε φανερὸν ὅτι τὰ δυνάμει ὄντα εἰς ἐνέργειαν ἀγόμενα εὑρίσκεται: αἴτιον δὲ ὅτι ἡ νόησις ἐνέργεια: ὥστ᾽ ἐξ ἐνεργείας ἡ δύναμις, καὶ διὰ τοῦτο ποιοῦντες γιγνώσκουσιν, ὕστερον γὰρ γενέσει ἡ ἐνέργεια ἡ κατ᾽ ἀριθμόν.[227]

This passage is not easy to translate; sentences often omit subjects or objects; the main terms such as διαγράμματα or ἐνέργεια can be read in more than one way, and the logical links between the propositions which constitute the whole argument are not perspicuous. Nevertheless, it is obvious that the passage supports one possible interpretation of the previous passage's claim, that 'existing matter-like' means 'existing

[227] *Met.*, XI, 9, 1051a21-33. Aristotle cites two geometrical theorems in this passage. The first is that the internal angles of a triangle are equal to two right angles. Euclid's proof can be found at *Elements* I. 32. But there is also a non-Euclidean proof for this therorem, which is regarded as Pythagorean (see Heath (1956) Vol. I, pp. 317-321). Makin argues that the proof Aristotle has in mind here is the non-Euclidean (see Makin (2006) pp.233-234). The second theorem to be proved is that any angle enclosed in a semicircle is a right angle. There are both Euclidean and non-Euclidean proofs for this theorem too (for the non-Euclidean proof, see Burnyeat (1984) p. 184. For the Euclidean, see *Elements*, III, 31). It is not clear which proof Aristotle represents here. Some commentators consider it his own (see Ross (1924) Vol, II, pp. 270-271; Heath (1956) Vol. II, pp. 63-64).

in potentiality.' I will deal with this interpretation later.[228] For now it is enough to point out that this passage embodies a constructivist aspect of Aristotle's view on mathematics, and that such features are rarely compatible with naïve realism. [229]

The Greek, 'διαγράμματα', in the first sentence, can be translated either 'geometrical propositions', 'geometrical figures' or 'geometrical constructions'. Only the third option, though, makes sense of the next sentence (εἰ δ' ἦν διῃρημένα, φανερὰ ἂν ἦν: νῦν δ' ἐνυπάρχει δυνάμει). The subject of this clause is omitted, but it is reasonable to take 'διαγράμματα' as its subejct. However, it makes sense neither to say that 'geometrical propositions had been divided (ἦν διῃρημένα)'; nor 'geometrical figures are in there in potentiality' since, in the given two examples, the proofs begin with fully actualized geometrical figures, i.e., a triangle and semicircle.

'ἐνεργείᾳ' in the first sentence could also mean either 'act (or activity)' or 'actuality'. Either way, though, sustains the same inference that mathematical objects are mentally constructed. If we translate the term as 'act', the first sentence can be rendered, 'geometrical constructions are discovered by act[s] because they (mathematicians) discover them by dividing.[230] The act in question, then, is that of dividing, that is, we discover geometrical constructions by dividing geometrical figures. For instance, in the second proof, the geometrical construction is obtained by dividing a semicircle by

[228] See Chapter Four, §3.

[229] For a full analysis of this passage, see Makin (2006), pp.232-246. See also Ross (1924) Vol. II, pp, 268-273 and Buryeat (1984) pp. 147-154.

[230] This is Ross' translation. See, Ross (1924) Vol. II, pp. 268-269.

straight lines. But in the last sentence Aristotle identifies 'ἐνέργεια' with thought (ἡ νόησις ἐνέργεια). This suggests that the dividing takes the form not of the physical drawing of lines with pen and paper, but rather consists in a purely mental activity. In fact, what a geometer talks about is not the physical drawing as such, for this never rigorously satisfies the definitions of geometry; and it is furthermore quite possible to do geometry without actual drawing. Aristotle shows his agreement with this by saying:

> Geometers do not suppose falsehoods, as some people have asserted, saying that you should not use falsehoods but that geometers speak falsely when they say that a line which is not a foot long is a foot long or that a drawn line which is not straight is straight. But geometers do not conclude anything from there being this line which he himself has described, but what is shown through them.
>
> οὐδ᾽ ὁ γεωμέτρης ψευδῆ ὑποτίθεται, ὥσπερ τινὲς ἔφασαν, λέγοντες ὡς οὐ δεῖ τῷ ψεύδει χρῆσθαι, τὸν δὲ γεωμέτρην ψεύδεσθαι λέγοντα ποδιαίαν τὴν οὐ ποδιαίαν ἢ εὐθεῖαν τὴν γεγραμμένην οὐκ εὐθεῖαν οὖσαν. ὁ δὲ γεωμέτρης οὐδὲν συμπεραίνεται τῷ τήνδε εἶναι γραμμὴν ἣν αὐτὸς ἔφθεγκται, ἀλλὰ τὰ διὰ τούτων δηλούμενα.[231]

This passage has perplexed commentators who adhere to the naïve realistic interpretation. But once we accept that Aristotle takes mathematical objects to be mental, the message of the whole paragraph is so clear as not to require further commentary: Do not confuse a mathematical object with its physical representation.

[231] *Apo.*, I, 10, 76b39-77a3.

If, in the preceding contentious passage, we translate 'ἐνέργεια' as 'actuality,[232] the implication is even more obvious. Since geometrical constructions are found in actuality (τὰ διαγράμματα ἐνεργείᾳ), and the actuality is thought, it follows that geometrical constructions are found in thought. Thus, on either translation, the conclusion of the passage will be that geometrical constructions exist in thought. This inference is assured by Aristotle's own statement:

> But of what is already a combined whole, for instance, of this circle, or of any of sensible or intelligible particulars—I mean by intelligible circles mathematical circles and by sensible circle circles of bronze or wood—there is no definition; but these are known by thought or perception.
>
> τοῦ δὲ συνόλου ἤδη, οἷον κύκλου τουδὶ καὶ τῶν καθ᾽ ἕκαστά τινος ἢ αἰσθητοῦ ἢ νοητοῦ λέγω δὲ νοητοὺς μὲν οἷον τοὺς μαθηματικούς, αἰσθητοὺς δὲ οἷον τοὺς χαλκοῦς καὶ τοὺς

[232] According to Aristotle, the original meaning of 'ἐνέργεια' is activity (*Met*., IX, 1, 1046a1-2; 3, 1047a30-32; 8, 1050a21-22). Makin points out that Aristotle uses the term in that sense in one of earlier works, the *Protrepticus* (see Makin (2006) p.xxviii). One of Aristotle's motives in introducing 'ἐντελέχεια' as a new technical term might be to avoid the multiple meanings of ἐνέργεια. Nevertheless, there are reasons to prefer a translation of 'actuality' to 'activity' in the above passage (1051a21-33). First, except in a few earlier works, Aristotle consistently uses 'ἐνέργεια' as a technical term to designate actuality. Second, even after introducing the new term, 'ἐντελέχεια,' Aristotle tends to reserve 'ἐνέργεια' for 'actuality.' For instance, in *Met*., IX, where he develops the concept of potentiality in contrast with actuality, he mainly uses the term 'ἐνέργεια' to signify actuality, using 'ἐτελέχεια' only six times. Finally, it is unnatural to read 'ἐτελέχεια' as 'act' in the final sentence, "αἴτιον ... ἀριθμόν." So, unless we suppose that Aristotle uses the same term in the same passage according to two different meanings, it would seem to be more consistent to read 'ἐνέργεια' to mean 'actuality' throughout the passage. For an account of Aristotle's usage of the term, 'ἐντελέχεια,' see Makin (2006) p. xxvii-xxx.

> ξυλίνους τούτων δὲ οὐκ ἔστιν ὁρισμός, ἀλλὰ μετὰ νοήσεως ἢ αἰσθήσεως γνωρίζονται.[233]

Aristotle, here, clearly distinguishes the mathematical circle from the perceptible; while the former is intelligible (κύλος νοητοῦ),[234] the latter is perceptible (κύλος αἰσθητοῦ). This is in accord with the preceding conclusion that the geometrical constructions exist in thought.

Further, if those geometrical constructions are *constructed* by thought, we can also argue that geometrical figures in general are able to be constructed by thought. The principle of constructing a geometrical construction and a figure is just the same, i.e., combining straight lines and circles. In the second example of geometrical construction, in fact, two triangles must be posited, suggesting that Aristotle supposes these to be available by way of mental activity.

If one were unaware of this idealistic aspect of Aristotle's philosophy of mathematics, the following paragraph, noted earlier, would hardly appear Aristotelian:

> Again, if it is not as they say, what sort of thing must the mathematician be supposed to deal with? Clearly not with the things in this world; for none of these is the sort of thing which the mathematical sciences investigate.
>
> εἰ δ' αὖ μὴ ἔστιν ὡς λέγουσι, περὶ ποῖα θετέον πραγματεύεσθαι τὸν μαθηματικόν; οὐ γὰρ δὴ περὶ τὰ δεῦρο:

[233] *Met.*, VII, 10, 1036a2-5.

[234] 'Intelligible' is the translation of the Greek 'νοητοῦ', which was given as thinking (νόηις) in the previous passage.

> τούτων γὰρ οὐθέν ἐστιν οἷον αἱ μαθηματικαὶ ζητοῦσι τῶν ἐπιστημῶν.[235]

But if it is allowed that Aristotle grants that mathematical objects are constructed by the mind, it makes perfect sense for him to say that the objects of mathematics are not in this world.

6. Three Issues concerning the Interpretation of Aristotle's Theory of Mathematics

It now seems that there as many passages of Aristotle rebutting a naive realistic view of mathematics as there are articulating it. As a naïve realist Aristotle maintains that:

(1) Mathematical objects do not exist separated from sensible substances.

and

(2) Mathematical objects are obtained from sensible by abstraction.

From (1) and (2), as I have shown, we can further infer

(3) Mathematics studies certain aspects or properties of sensible objects.

[235] *Met.*, XI, 1, 1059b9-12.

But we also have noted the idealistic aspect of his concept of mathematics, that is, he also argues that:

(4) Mathematical objects are not in sensible objects

(5) Mathematical objects exist only as matter.

(6) The actuality of a geometrical construction is a kind of thought.

and

(7) Geometrical objects are constructed in thought.

All the claims, (1) thorough (7), are taken directly from Aristotle's text. But those two groups of claims (1)-(3) and (4)-(7), hardly seem compatible with each other. For example, we have seen that (1) and (2) imply that (3) the objects of mathematics are properties of sensible objects. (3) is also consistent with idea that the objects of applied mathematics are certain properties of sensible things, e.g., the objects of astronomy are the mathematical properties of the movements of heavenly bodies in the sky. Yet it has been also shown that (4) implies that:

(8) Mathematical objects are *not* properties of sensible objects.

(5), (6), and (7) also support (8). But (8) contradicts (3). This means that Aristotle's mathematical text contains or implies propositions that are incompatible one another. We may call this the internal consistency problem.

At this point, it seems legimate to throw doubt on whether Aristotle really had one unified theory of mathematics; I have proposed the hypothesis he has two different theories, one of which was developed later, based on the other, but ultimately diverging from it to the point that the two accounts became incompatible.[236] If my hypothesis is right, there can be no interpretation which accommodates both groups of claims. Thus, instead of pronouncing on the unity of Aristotle's mathematical theory in this sense, I will review the interpretations suggested by other commentators; if it turns out that none succeed in resolving the internal consistency problem, we will have a good reason to believe that Aristotle developed two different and incompatible theories.

It is appropriate at this stage, though, to clarify once again the problems which Aristotle's philosophy of mathematics involves. First, Aristotle explains that mathematical objects are something obtained from sensible objects by abstraction; but this raises the precision problem, since many geometrical figures are not instantiated in sensible objects. Secondly, we also have seen that Aristotle's mathematical texts seem to contain two opposite positions, i.e., naïve realism and idealism. This throws doubt on the consistency of his philosophy of mathematics. Besides those two problems, there is one further issue to be considered in examining commentators' interpretations. Any position on mathematics set out as an interpretation of Aristotle should be consistent with his scientific realism, i.e., mathematical objects as the objects of a science should exist mind-independently just like the objects of other sciences. But the question whether mathematical entities exist can be asked in different senses insofar as different

[236] See Chapter Three, §4.

metaphysics invoke different senses of being. For instance, there is a gap between Aristotle's ontology and that of modern physicalism. Aristotle's ontology is more generous: in his ontology, for instance, not only all catergories of beings in actuality[237] but also potential beings[238] are regarded as something existent. Thus, the question of the reality of mathematical entities for Aristotle involves two different issues: One, whether mathematical objects fit into his own ontological framework as existents; the other, whether those objects can be seen as being existent from the other metaphysical perspectives (such as physicalism, for instance). I will deal with only the first; my concern is whether the truth of mathematics can be accounted for from within Aristotle's own metaphysics. If so, this strengthens the appeal of his metaphysics. Thus, in seeking to show that Aristotle's philosophy of mathematics is congruent with his scientific realism, we will examine what place there is for mathematical objects in his ontology.

In order for any interpretation of Aristotle to be complete, we need to provide a solution to these three problems. This means that the three problems will serve as standards to be met in the evaluation of any purported unifying interpretation. In fact, commentators have struggled with these problems directly or indirectly. In the next chapter, I assess some interpretations of Aristotle' theory of mathematics on the basis of whether they deal with these issues successfully.

[237] See *Met*., XIV, 2, 1089a5-7. cf. Frede (1987) pp. 72-73; *Met*., XIV, 2, 1089a7-9.

[238] In general, in Aristotle potentiality (δύναμις) is distinguished from mere possibility; an actual being is ontologically prior to a potential being. See Witt (2003) pp. 79 and 81. However, it is doubtful whether such a notion of potentiality is applicable to mathematicals. This issue will be discussed in Chapter Four, §4. For Aristotle's actualism, see Chapter Four, §3.

Chapter Four

Interpretations of Aristotle's Philosophy of Geometry

The last chapter identified three main issues to be considered in any interpretation of Aristotle's philosophy of mathematics. The first is the precision problem: Aristotle's theory of abstraction implies that mathematical objects are obtained from sensible substances; but at the same time sensible particulars lack any properties that might satisfy the definitions of geometrical figures. The second issue concerns the internal consistency of Aristotle's views. Despite Aristotle's commitment to mathematical naïve realism, some passages in Aristotle's texts imply rather some form of mathematical idealism. Finally, the idealistic aspect of Aristotle's views raised the question whether his theory of mathematics can be made compatible with his scientific realism.

A number of commentators have offered interpretations aiming to resolve these problems. While these interpretations vary, depending on the aspects of Aristotle's view to which they attend, Mueller distinguishes three principal lines of interpretation summarized as follows: (M1) "Mathematical objects are physical objects which the mathematician studies by leaving out of account their mathematically irrelevant properties." (M2) "Mathematical objects are embodied in pure extension underlying physical objects." (M3) Mathematical objects exist only in the mind of the mathematician who reasons about triangles, angles, etc.[239] Mueller identifies Lear's view with (M1); he

[239] See Mueller (1990) p.464. See also White (1993) pp. 181-182.

himself endorses (M2); and thinks that (M3) is not explicitly represented in modern scholarship.[240]

Without taking issue with Mueller's distinction between these three main positions, I do not think that his comments on each are quite right. While (M1) is the main tenet of mathematical naïve realism, as we will see, Lear himself clearly states that his own position is a kind of fictionalism, i.e., it holds that a mathematical object is constructed out of elements abstracted from sensible objects. In fact, position (M1) is the one that receives support from only few modern commentators because of the first two issues mentioned above, namely, the precision problem and the internal consistency problem;[241] whereas (M3) would seem to approximate well to the view that Hintikka develops and that, I will argue, deserves more attention than it receives.

Since I have already examined the naïve realistic interpretation, I will pass over consideration of (M1), rather moving at once to examine Lear's actual position, namely that:

> (M1') The basic elements of geometrical figures, such as circles and straight lines, are abstracted from sensible objects, and all geometrical figures are constructed out of them.

[240] Mueller (1990) p. 465.

[241] Apostle interprets Aristotle's philosophy of mathematics as naïve mathematical realism. He argues that, for Aristotle, mathematics is the science of quantities of sensible objects. See Apostle (1952) pp. 14 and 28.

In addition to this, I will consider another possible position, which is based on the assumption of (M2):

(M2') Mathematical objects exist in potentiality.

The four positions under review in this chapter, then, will be (M1'), (M2), (M2'), and (M3).

Unlike the naive realistic interpretation, all of these interpretations admit that the existence of mathematical objects is mind-dependent in some way. This admission is unavoidable if we are to accommodate the idealistic aspect of Aristotle's theory of mathematics. Keeping in mind the three issues raised above, let us begin by examining position (M1').

1. Realistic Constructivism

In his paper, 'Aristotle's Philosophy of Mathematics', Lear suggests a solution to the precision problem. Along with most modern Aristotelian scholars, Lear agrees that Aristotle's abstraction is a conceptual procedure; it filters out predicates that happen to be true of an object under a ceratin aspect.[242] For instance, if geometry studies a bronze isosceles triangle *qua* triangle, the *qua*-operator filters out 'being brazen' and 'being isosceles'. But, as we have seen, if geometry is the study of those properties of sensible

[242] Lear (1982) p. 168.

objects that remain after such conceptual filtration, the precision problem becomes unavoidable. Sensible objects cannot precisely instantiate forms of geometrical figures. While recognizing this problem,[243] Lear argues that it is misconceived to suppose that a first claim, that geometrical objects are some properties of sensible particulars, needs to involve a second claim, that every geometrical figure has a physical instantiation. Rather, only the basic geometrical elements, out of which more complicated geometrical figures can be constructed, require physical instantiation.[244] For example, to construct a triangle, we need only three straight lines; so if we can obtain some minimal form of straightness (i.e., a straight line) out of physical objects by abstraction, we can construct a triangle.

How many elements will we need, then, in order to construct every geometrical figure? In Euclidian geometry, all geometrical objects except points and curved lines can be made from straight lines, circles, and spheres. But since, in Aristotle's view, points exist only as the extremities of lines[245] and a curve is no more than a combination of a straight line and a circle, there is no need to posit points or curved lines apart from circles and straight lines.

It turns out that the thesis that every geometrical figure has its physical counterpart is neither a philosophically plausible claim, nor one that Aristotle consistently holds. As the previous chapter made clear, though, Aristotle does believe that there are perfect physical instantiations of circularity, straightness, and sphericity. Thus, Lear

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[243] *Ibid*., p. 179.
[244] *Ibid*., p. 181.
[245] This is also the definition of a point in Elements. See *Elements*, I, Def. 3. One of the features of Aristotle's philosophy of mathematics is that it is based on the practice of mathematics in his era.

argues, we can abstract every basic element of geometry from physical objects, out of which all geometrical figures can be constructed.

Lear's interpretation has several merits over that of the naïve realists'. First, it can avoid the problem of precision. Since each geometrical figure is constructed from the basic geometrical elements, there is no need for every geometrical figure to be instantiated i.e. in physical form individually. But geometry can still be said to be a study of sensible physical objects; the basic geometrical elements out of which geometrical objects are made are acquired from sensible particulars through abstraction. Secondly, this interpretation provides a partial solution to the internal consistency problem. We saw that Aristotle's theory of geometry contains seemingly incompatible assertions. For instance, on the one hand, Aristotle argues that geometrical figures are constructed and are not *in* sensible objects. On the other, he maintains that geometry is a study concerned with sensible objects, and that mathematical objects are properties of sensible objects obtained by means of abstraction. But even if we acknowledge Aristotle's second claim, there is still a sense in which he can say that a geometrical figure is constructed, since it is constructed out of basic elements; the figure is not itself in a sensible, though its elements are. Nevertheless, geometry remains a study concerned with sensible objects because (i) it studies such properties of sensibles as circularity and straightness, and (ii) all geometrical figures are composed of these basic elements obtained from sensible things.

In addition, this interpretation also provides a possible explanation for Aristotle's perplexing statement that mathematical objects exist only as matter.[246] One meaning of 'matter' in Aristotle's work is the 'parts of a whole'. In *Met.,* VII, dealing with the question of priority in formulae, he uses 'matter' to designate the parts into which the whole is divided. He says that:

> But the formula of a right angle is not divided into the formula of the acute, but [rather] the formula of the acute includes that of the right angle; for one who defines the acute uses the right; for acute is less than a right angle. The circle and the semicircle also hold the same relation; for the semicircle is defined by the circle and the finger by the whole body; for a finger is such a part of a man. Thus, all parts which are matter, into which a thing is divided as into matter, are posterior.
>
> ὁ δὲ τῆς ὀρθῆς λόγος οὐ διαιρεῖται εἰς ὀξείας λόγον, ἀλλ᾽ <ὁ> τῆς ὀξείας εἰς ὀρθήν: χρῆται γὰρ ὁ ὁριζόμενος τὴν ὀξεῖαν τῇ ὀρθῇ: ἐλάττων γὰρ ὀρθῆς ἡ ὀξεῖα. ὁμοίως δὲ καὶ ὁ κύκλος καὶ τὸ ἡμικύκλιον ἔχουσιν: τὸ γὰρ ἡμικύκλιον τῷ κύκλῳ ὁρίζεται καὶ ὁ δάκτυλος τῷ ὅλῳ: τὸ γὰρ τοιόνδε μέρος ἀνθρώπου δάκτυλος. ὥσθ᾽ ὅσα μὲν μέρη ὡς ὕλη καὶ εἰς ἃ διαιρεῖται ὡς ὕλην, ὕστερα.[247]

This meaning of 'part' is found in another passage where Aristotle distinguishes five senses of parts,[248] one of which is 'the elements' into which the whole is divided, or of which it consists. In the case of the bronze sphere or the bronze cube, for instance, he takes both the bronze and the characteristic angles as parts, further applying the distinction between parts and the whole to formulae. The elements in the formula which

[246] *Met*., XIII, 3, 1078a28-31
[247] *Met.* VII, 10, 1035b4-12
[248] *Met*. V, 24, 1023b12-35.

explains a thing are regarded as parts of the whole formula. In either meaning, the basic geometrical elements may be regarded as parts of geometrical figures. For instance, a triangle consists of three straight lines; it is defined as being a plane figure contained by three straight lines.[249] On this understanding, the basic geometrical elements may be understood as a kind of matter, in other words the constitutive matter of a geometrical figure. Since what we can obtain from sensible objects by abstraction are only the basic elements of geometrical figures, it makes sense to say that mathematical objects exist only as matter.

In spite of its advantages, Lear's interpretation has two main problems. The first is that it is not necessary to acquire the basic geometrical elements from sensible objects by abstraction in order to construct geometrical figures; if we can construct a figure, the basic element can be constructed as well. The second is that the interpretation turns Aristotelian mathematical objects into fictional entities. To assimilate Aristotle's position to fictionalism is problematic because: (i) fictionalism is not compatible with Aristotle's scientific realism, and (ii) assuming fictionalism, there would be no way to account for the truth of mathematics given Aristotle's own theory of truth. Let us consider each problem in turn.

Lear's account of Aristotle's theory of mathematics can be summarized in the following three theses:

[249] *Elements*, I, Def. 19.

(1) It is not possible to acquire every geometrical form from sensible objects by means of abstraction,

(2) We can obtain the basic geometrical elements such as circles and straight lines through abstraction,

and

(3) It is possible to construct every geometrical figure from the basic elements.

But if (3) is true, why can we not construct the basic elements as well? Although the definition of triangle includes the concepts of straight line and figure, neither of those concepts contains the concept of triangle. Thus, if (3) is admitted and triangularity cannot be obtained from sensible objects by abstraction, it should be the case that the concept or form of a triangle is created by or imposed on the elements by our mind. If our mind has such a capacity, then there is no reason to believe that we cannot also create the concepts of basic geometrical elements such as 'line', 'circle', and 'point', as well. For instance, a *straight line* is defined as a line which lies evenly with the points on itself.[250] Thus, if we know the concepts of line and point, we can also have the concept of a straight line. But since 'line' and 'point' are respectively defined as 'breadthless length' and 'that which has no part,' the concept of a 'triangle' only requires us to know the concepts of 'breadth,' 'length,' and 'part.' If this is so, Lear's effort to accommodate Aristotle's naïve realism in his interpretation by showing that there exist perfect physical

[250] *Elements*, I, Def. 4.

instantiations of the basic geometrical elements, out of which every geometrical figure can be constructed, is pointless; that fact does not make any difference.

The second objection is more crucial. Lear argues that a geometrical figure is constructed from the elements abstracted from sensible objects. However, if a geometrical object like a triangle is constructed (or created at any rate), it is hard to see in what sense it exists. Although being has broader meaning in Aristotle's metaphysics than in modern physicalists', there seems to be no room for mentally constructed things in his ontology. Suppose that we mentally imagine a pink elephant. Suppose, further, that the appearance of that pink elephant is obtained from an actual elephant by abstraction and its color pink is also an actual color. Then, Lear would say that the elephant is composed of elements abstracted from actual sensible objects. Should we then say that the pink elephant exists? We can note here that most mythical animals, such as unicorns and chimeras, are composed of elements obtained from actual animals. If we cannot say that the pink elephant or any mythical animal exists, then, a triangle cannot be said to exist either. The pink elephant, the unicorn, the chimera and the triangle are all mentally constructed out of elements abstracted from the actual world. Thus, Lear's interpretation makes Aristotle's mathematical objects fictional entities. It is interesting that Lear himself recognizes this implication.

> There may be no purely geometrical objects, but they are a useful fiction, because they are an obvious abstraction from features of the physical world…talk of nonphysical mathematical objects is fiction, one that may

> be convenient and should be harmless if one correctly understands mathematical practice.[251]

Lear even compares Aristotle's position with Field's fictionalism, and argues that they share the feature of not positing mathematical entities beyond physical objects.[252]

However, there are good reasons to resist the fictionalist interpretation. First of all, it is contradictory with Aristotle's scientific realism. We saw that, for Aristotle, knowledge is always of something existent; he agrees with Plato that the objects of every science should exist. Indeed, it is one of main tenets of Aristotle's philosophy of mathematics that mathematical objects exist.[253] This point has been discussed sufficiently in the previous chapter. Thus, if there is no knowledge of non-being, and mathematical objects are fictional entities, there can be no mathematical knowledge. As Apostle points out, for Aristotle 'sciences are concerned with what exists and not with

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[251] Lear (1982) pp. 188 and 191.

[252] *Ibid.*, pp. 187-191. For Field's fictionalism, see Field (1980); Field (1982); Field (1984). For Field's influence and the responses of his fictionalism, see Irvine (1990) p. xxii.

[253] Aristotle says: "It is also absolutely true say that the objects of mathematics exist and those things which mathematicians are talking about (καὶ τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν, καὶ τοιαῦτά γε οἷα λέγουσιν) (*Met.*, XIII, 2, 1077b32-34)."; "But since they could not exist in sensibles either, it is obvious that they either do not exist at all or exist in some way and for this reason they do not exist absolutely. For 'to exist' is said in many ways (ἐπεὶ δ' οὐδ' ἐν τοῖς αἰσθητοῖς ἐνεδέχετο αὐτὰ εἶναι, φανερὸν ὅτι ἢ ὅλως οὐκ ἔστιν ἢ τρόπον τινὰ ἔστι καὶ διὰ τοῦτο οὐχ ἁπλῶς ἔστιν: πολλαχῶς γὰρ τὸ εἶναι λέγομεν) (*Ibid.*, 2, 1077b14-17)."; "About mathematical objects, thus, we have said…that they exist and in what sense they exist (Περὶ μὲν οὖν τῶν μαθηματικῶν, ὅτι τε ὄντα ἐστὶ καὶ πῶς ὄντα...εἰρήσθω) (*Ibid.,* 4, 1078b7-9) ."

what does not exist';[254] so, if the objects of mathematics are fictions or non-beings, mathematics is not a science in the Aristotelian sense. But mathematics is Aristotle's prime example of a science.

Second, if mathematical objects are fictional, there is way no way to explain mathematical truth in a way that is consistent with Aristotle's general concept of truth. The answer to the question of how mathematics can be true in Aristotle, if the objects of its study do not exist, will depend on what kind of theory of truth Aristotle is taken to hold. But it is acknowledged that concerning the concept of truth, Aristotle is faithful to 'what is (τὸ ὂν)':

> This will be plain if we first define what truth and falsehood are: to say that what is is not, or what is not is, is false, while to say that what is that it is, or of what is not that it is not, is true; and therefore he who says that a thing is or thing is not, will say either what is true or what is false.
>
> δῆλον δὲ πρῶτον μὲν ὁρισαμένοις τί τὸ ἀληθὲς καὶ ψεῦδος. τὸ μὲν γὰρ λέγειν τὸ ὂν μὴ εἶναι ἢ τὸ μὴ ὂν εἶναι ψεῦδος, τὸ δὲ τὸ ὂν εἶναι καὶ τὸ μὴ ὂν μὴ εἶναι ἀληθές, ὥστε καὶ ὁ λέγων εἶναι ἢ μὴ ἀληθεύσει ἢ ψεύσεται.[255]

This paragraph embraces the basic idea of a correspondence theory of truth, i.e., that truth is a certain relation between a saying (λέγειν) and reality (τὸ ὂν). But, since 'εἶναι' is multiply ambiguous in Greek, in particular between existential, verdical and

[254] Apostle (1952) p.11. cf. *Apo*, I, 10, 76b17-23.
[255] *Met.*, IV, 7, 1011b25-8. The same idea of truth is also expressed in Plato (*Cratylus* 385b2 and *Sophist* 263b).

predicative senses, 'τὸ ὂν' could be variously translated into 'that which exists', 'that which is the case', or 'is so-and-so'. So the significance of τὸ ὂν remains elusive. Nevertheless, in the *Categories* where Aristotle talks of "things" underlying an affirmative or negative statement, we find what on the part of τὸ ὂν is responsible for the truth of some statement.[256] He says:

> For affirmation is an affirmative statement and a negation a negative statement, but none of the things underlying affirmation and negation is a statement. But these are also said to be opposed to one another as affirmation and negation, for in these cases, too, the way of opposition is the same. For in the way an affirmation is opposed to a negation, e.g., 'he is sitting'—he is not sitting', so are opposed also the actual things underlying each, his sitting—his not sitting.
>
> ἡ μὲν γαρ κατάφασις λόγος ἐστὶ καταφατικὸς καὶ ἡ ἀπόφασις λόγος ἀποφατικός, τῶν δὲ ὑπὸ τὴν κατάφασιν ἢ ἀπόφασιν οὐδέν ἐστι λόγος. λέγεται δὲ καὶ ταῦτα ἀντικεῖσθαι ἀλλήλοις ὡς κατάφασις καὶ ἀπόφασις: καὶ γὰρ ἐπὶ τούτων ὁ τρόπος τῆς ἀντιθέσεως ὁ αὐτός: ὡς γάρ ποτε ἡ κατάφασις πρὸς τὴν ἀπόφασιν ἀντίκειται, οἷον τὸ κάθηται οὐ κάθηται, οὕτω καὶ τὸ ὑφ' ἑκάτερον πρᾶγμα ἀντίκειται, τὸ καθῆσθαι μὴ καθῆσθαι.[257]

Since what underlies the statement "he is sitting" is the fact that he is sitting, it can be said that, for Aristotle, truth is the correspondence between a statement and a fact.

[256] *Cat.* 10, 12b11; 14, 14b14.
[257] *Cat*. 10, 12b6-16.

Although truth is a relation between two terms, a statement and a fact, actual facts are prior to statements' being true:

> For if there is a man, the statement wherein we say that there is a man is true, and, conversely, if the statement wherein we say that there is a man is true, there is a man. But the true statement is in no way the cause of the actual thing's existence, whereas the actual thing seems in some way the cause of the statement's being true; for the statement is called true or false by virtue of the actual thing's existing.
>
> εἰ γὰρ ἔστιν ἄνθρωπος, ἀληθὴς ὁ λόγος ᾧ λέγομεν ὅτι ἔστιν ἄνθρωπος: καὶ ἀντιστρέφει γε, εἰ γὰρ ἀληθὴς ὁ λόγος ᾧ λέγομεν ὅτι ἔστιν ἄνθρωπος, ἔστιν ἄνθρωπος: ἔστι δὲ ὁ μὲν ἀληθὴς λόγος οὐδαμῶς αἴτιος τοῦ εἶναι τὸ πρᾶγμα, τὸ μέντοι πρᾶγμα φαίνεταί πως αἴτιον τοῦ εἶναι ἀληθῆ τὸν λόγον: τῷ γὰρ εἶναι τὸ πρᾶγμα ἢ μὴ ἀληθὴς ὁ λόγος ἢ ψευδὴς λέγεται.[258]

Thus, Aristotle maintains that "statements are true according to how the actual things are (ὁμοίως οἱ λόγοι ἀληθεῖς ὥσπερ τὰ πράγματα)."[259] On the basis of this concept of truth, it seems obvious that if mathematical objects do not exist or are fictional, mathematics cannot be true at all.[260] If triangles do not exist, for instance, there cannot

[258] *Ibid*., 12, 14b14-22.

[259] *Int.,* 9, 19a.33.

[260] Aristotle's position on the truth values of statements of non-being is not clear. He seems to argue that there is no true statement of non-being, when he says that "But it might especially very well seem that such a thing occurs in the case of contraries said in combination, for 'Socrates is well' being contrary to 'Socrates is sick'. But not even with these is it always for one to be true and the other false. For if Socrates exists one will be true and one false, but if he does not exist, both will be false; neither 'Socrates is sick' nor 'Socrates is well' will be true if Socrates himself does not exist all. As for possession and privation, if he does not exist at all, neither is true, but if he does, not always one or the other is true. For 'Socrates has sight' is opposed to 'Socrates is

be any actual facts about triangles; and if there are no such facts, no statement about triangles can be true since there is no fact to correspond to it.

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blind' as possession to privation; and if he exists it is not necessary for one or the other to be true or false (when he has not yet grown to have it both are false), while if Socrates does not exist at all, then both are again false, both 'he has sight' and 'he is blind' (οὐ μὴν ἀλλὰ μάλιστα δόξειεν ἂν τὸ τοιοῦτο συμβαίνειν ἐπὶ τῶν κατὰ συμπλοκὴν ἐναντίων λεγομένων, τὸ γὰρ ὑγιαίνειν Σωκράτη τῷ νοσεῖν Σωκράτη ἐναντίον ἐστίν, ἀλλ᾽ οὐδ᾽ ἐπὶ τούτων ἀναγκαῖον ἀεὶ θάτερον μὲν ἀληθὲς θάτερον δὲ ψεῦδος εἶναι: ὄντος μὲν γὰρ Σωκράτους ἔσται τὸ μὲν ἀληθὲς τὸ δὲ ψεῦδος, μὴ ὄντος δὲ ἀμφότερα ψευδῆ: οὔτε γὰρ τὸ νοσεῖν Σωκράτη οὔτε τὸ ὑγιαίνειν ἀληθὲς αὐτοῦ μὴ ὄντος ὅλως τοῦ Σωκράτους. ἐπὶ δὲ τῆς στερήσεως καὶ τῆς ἕξεως μὴ ὄντος γε ὅλως οὐδέτερον ἀληθές, ὄντος δὲ οὐκ ἀεὶ θάτερον ἀληθές: τὸ γὰρ ὄψιν ἔχειν Σωκράτη τῷ τυφλὸν εἶναι Σωκράτη ἀντίκειται ὡς στέρησις καὶ ἕξις, καὶ ὄντος γε οὐκ ἀναγκαῖον θάτερον ἀληθὲς εἶναι ἢ ψεῦδος, ὅτε γὰρ μήπω πέφυκεν ἔχειν, ἀμφότερα ψευδῆ, μὴ ὄντος δὲ ὅλως τοῦ Σωκράτους καὶ οὕτω ψευδῆ ἀμφότερα, καὶ τὸ ὄψιν αὐτὸν ἔχειν καὶ τὸ τυφλὸν εἶναι(*Cat*. 10, 13b12-27))."

At the same time, Aristotle accepts that there can be true statements of non-being:, "But with an affirmation and negation, whether he exists or not, one will always be false and the other true. For 'Socrates is sick' and 'Socrates is not sick': If he exists it is clear that one or the other of them will be true or false, and similarly if he does not exist; if he does not exist 'he is sick' is false but 'he is not sick' true. Thus it would be distinctive of these alone that one of two is always true or false, so far as affirmation and negation are opposed (ἐπὶ δέ γε τῆς καταφάσεως καὶ τῆς ἀποφάσεως ἀεί, ἐάν τε ᾖ ἐάν τε μὴ ᾖ, τὸ μὲν ἕτερον ἔσται ψεῦδος τὸ δὲ ἕτερον ἀληθές: τὸ γὰρ νοσεῖν Σωκράτη καὶ τὸ μὴ νοσεῖν Σωκράτη, ὄντος τε αὐτοῦ φανερὸν ὅτι τὸ ἕτερον αὐτῶν ἀληθὲς ἢ ψεῦδος, καὶ μὴ ὄντος ὁμοίως: τὸ μὲν γὰρ νοσεῖν μὴ ὄντος ψεῦδος, τὸ δὲ μὴ νοσεῖν ἀληθές. ὥστε ἐπὶ μόνων τούτων ἴδιον ἂν εἴη τὸ ἀεὶ θάτερον αὐτῶν ἀληθὲς ἢ ψεῦδος εἶναι, ὅσα ὡς κατάφασις καὶ ἀπόφασις ἀντίκειται (*Ibid*., 13b27-35))." Since the exclusive middle is posited as an axiom in Aristotle's logical system, for any proposition, P*a*, if P*a* is not true, not-P*a* must be true. Thus, if a sentence '*a* is P' is false, its contradictory, '*a* is not P' must be true even if a does not exist.

Although, in a sense, there can be a true statement of non-being, it is noteworthy that Aristotle never allows true affirmative statements of non-being; only negative statements are taken as examples of true statements of non-being. Thus, a possible interpretation would be that for Aristotle if *a* does not exist, any affirmative statement of *a* is false, and a negative statement can be true only when its contradictory is false.

These considerations, in effect, render inappropriate Lear's comparison of Aristotle's view to Field's fictionalism. Aristotle's philosophy of mathematics starts out from a quite different motivation from that of the fictionalists. Aristotle' problem was how to account for the truth of mathematics without positing Platonic entities within his metaphysics. In contrast, Field's strategy is to avoid any ontological commitment to mathematical objects, by denying that mathematics is true.[261] Field insists that there is no need to have a model for mathematics since the universal applicability of mathematics can be explained without mathematics' being true. Thus, to assimilate Aristotle's view to Field's fictionalism amounts to saying that Aristotle's philosophy of mathematics does not show that mathematics is true after all. This is exactly the opposite of Aristotle's position.

Field argues that the utility of mathematics can be explained in terms of '*conservativeness*.' For Field, mathematics is conservative in the sense that "any inference from nominalistic premises to a nominalistic conclusion that can be made with the help of mathematics could be made (usually more long-windedly) without it."[262] Thus, fictionalism can be seen as a reductionists' program which attempts to reduce mathematical facts to physical facts, which is why fictionalism argues against the existence of mathematical objects. However, Aristotle not only accepts the existence of mathematical objects, but also maintains that some mathematical objects are neither

[261] One of the main tenets of fictionalism is that a theory can be reliable without being true. In that sense, factionalism is affiliated with instrumentalism. cf. Field (1980) pp. viii, 7.

[262] Field (1980), p. viii. For more discussion of mathematical *'conservativeness'*, see, *Ibid*., pp. 8-13.

sensible substances nor some of their properties. In that sense, it could be said that Aristotle admits that there exist mathematical facts irreducible to physical ones.

So far, three objections to Lear's fictionalistic interpretation have been considered: First, it is not compatible with Aristotle's scientific realism. In particular, Lear's fictionalism is contrary to Aristotle's commitment to the existence of mathematical objects. Secondly, if mathematical objects are fictional entities, there is no way to explain how mathematics can be true in Aristotle's metaphysics. Thirdly, Lear's comparison of Aristotle's view to Field's fictionalism is not appropriate for: (i) while the former aims to prove the truth of mathematics, the latter denies it, and (ii) Field argues that there is no mathematical fact beyond physical facts, whereas Aristotle rather argues that mathematical objects are *not* properties of sensibles; they are not in sensibles.

Unfortunately, Lear does not properly tackle the first objection; indeed, given his position, there seems to be no way for him to handle the conflict between his fictionalistic interpretation and Aristotle's commitment to mathematical realism. Lear does not seem to have considered the second problem associated with the third objection, either. We saw that Aristotle's view of geometry has two opposing tendencies: naïve realism and idealism. Many commentators on Aristotle's view of geometry tend to focus on one of these tendencies at the expense of the other, leading to tendentious readings that ignore large portions of the text at odds with their interpretations.[263] Lear is one such. In arguing for the existence of perfect physical instantiations of mathematical objects, he

[263] This happens more frequently when commentators interpret Aristotle's theory of number.

passes over the passages in which Aristotle denies the immanency of mathematical objects in sensible objects; at the same time, maintaining the fictionality of mathematicals in Aristotle, he ignores places where Aristotle explicitly states his mathematical realism.

Against the second objection, Lear argues that the fictionality of mathematical objects does not harm the truth of mathematics. He recognizes that, unlike fictionalism, Aristotle's theory aims at explaining the truth of mathematics:

> The great virtue of Aristotle's account is that Aristotle also takes great pains to explain how mathematics can be true. A conservative extension of physical theory needs not merely to be consistent; it can be true. Aristotle tries to show how geometry and arithmetic can be thought of as true, even though the existence of separated mathematical objects, triangles and numbers is harmless fiction.[264]

But how can a theory of fictional entities be true for Aristotle? Lear attempts to reconcile Aristotle's realism with his fictionalistic interpretation as follows:

> The key to explain the truth of a mathematical statement lies in explaining how it can be useful. Aristotle considered the truth of geometry to be useful because there are clear paths which lead one from the physical world to the world of geometrical objects. There may be no purely geometrical objects, but they are a useful fiction, because they are an obvious abstraction from features of the physical world…Thus, to explain the usefulness and applicability of mathematics we have to follow Aristotle and appeal far more strongly to the existence of a bridge between the physical world and the world of mathematical objects than we have to the fact that mathematics is a conservative extension of science.[265]

[264] Lear (1982) pp. 181-191.

[265] *Ibid.,* pp. 188-190.

Lear seems to argue here that mathematics is true because it is useful; he maintains that we can explain its truth by considering how it is useful. Further, he argues that mathematics is applicable to the physical world because there is a bridge between the physical world and mathematics. But what is this bridge, and how does its existence account for the truth of mathematics? For Lear:

> The important point is that direct links between geometrical practice and the physical world are maintained. Even in the case where the geometer constructs a figure in thought, one which perhaps has never been physically instantiated, that figure is constructed from elements which are direct abstractions from the physical world. Otherwise it will remain a mystery how, for Aristotle, geometry is supposed to be applicable to the physical world.[266]

Lear's stresses the 'direct links' between geometrical practice and the physical world that arise because geometrical figures are constructed out of elements obtained from sensible objects. To evaluate this argument, we can again consider the example of the pink elephant. We might say that there is a direct link between the pink elephant and the physical world; the description of its appearance is the description of an actual elephant. So, if there is a theory of a pink elephant, a part of the theory will seem to hold good of the actual world too. *Mutatis mutandis*, it might be argued that there is such a direct link

[266] *Ibid.*, p. 181.

between geometry and the physical world; geometry can be applied to the actual world to the extent that it deals with the actual properties of sensible objects.

But even if that were true, the geometrical knowledge that might be applicable to the actual world will be very limited; all theorems about geometrical objects other than circles, straight lines, spheres and points, would have nothing to do with the physical world; e.g., since there is no physical instantiation of a triangle, theorems about triangles would not true of the physical world. In this regard, a triangle would be better likened to a unicorn or chimera than to a pink elephant; while the pink elephant has the same shape as an actual elephant, there is no physical object whose shape is precisely the same as that of a triangle in geometry. Thus, the link between the physical world and geometrical objects is as weak as that between physical objects and a unicorn; every element of unicorns is obtained from physical objects by abstraction.

Moreover, whether or not there is such link between geometry and the physical world, it remains true that for Aristotle mathematical objects besides the basic geometrical elements do not exist in the actual world. And if they do not exist, there is no mathematical fact which could correspond to statements concerning the mathematical objects. For Aristotle, as we have seen, for a sentence, '*a* is P,' to be true, it is necessary that there exist *a* and P in the actual world. Thus, for Aristotle, if triangles do not exist, a theorem concerning triangles can be no more true than any description of a unicorn; e.g., ‘A triangle has the sum of two right angles’ would be no more true than ‘A unicorn has one horn’; and, as we have seen, statements like the latter must be false for Aristotle.[267]

Let us now consider Lear’s response to the third objection. As Lear is well aware, Field denies the truth of mathematics whereas Aristotle does not. But Lear argues that this difference is caused by Field’s mistaken view that mathematics can be true only if the objects of mathematics exist. In Lear’s view, mathematics can be true without the existence of its objects, so there is no need for fictionalism to give up mathematical truth:

> Field agrees with Aristotle that there are no separated mathematical objects, but thinks that for that reason alone mathematics is not true. From an Aristotelian perspective, Field looks overly committed to the assumptions of referential semantics: in particular, to the assumption that the way to explain mathematical truth via the existence of mathematical object. One can understand how mathematics can be true, Aristotle thinks, by understanding how it is applicable.[268]

Here again Lear appeals to his claim that mathematical truth consists in its applicability. But this claim is not Aristotle’s—in fact, it is not even Aristotelian. To be sure, Aristotle treats the applicability of mathematics as important. But he mentions it only to dispute mathematical Platonism, not to explain the truth of mathematics: If we posit mathematicals separated from sensible objects, there is no way to explain how the science

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[267] See Chapter Four, §1, n. 260.
[268] *Ibid*., p. 191

of mathematics is applicable to the sensible world.[269] For Aristotle, mathematics is applicable to the physical world because it is true of the physical world, not the other way around. And we saw that because of the precision problem, he has trouble explaining just how mathematics is true of the physical world. But he nowhere attempts to establish the truth of mathematics on the basis of its applicability. The idea that mathematics is true because it is applicable, does not correspond with Aristotle's theory of truth in general; Aristotle never mentions applicability as a necessary condition for the truth of a statement about an object, *a*, whereas, the existence of *a* is an essential prerequisite for its truth. Thus, although Lear criticizes Field for overly committing to "the assumption that the way to explain mathematical truth is via the existence of mathematical objects," for Aristotle, the existence of mathematical objects is a necessary condition for a mathematical statements' being true. And it is one of the main claims Aristotle makes about mathematical objects that they exist.

In recent years, there has been a tendency to interpret Aristotle as adopting a kind of fictionalism.[270] Thus, fictionalists argue that Aristotle's geometricals are fictional entities. For these interpreters, though, the fictionality of mathematical objects does not undermine the truth of mathematics in Aristotle.[271] Putting aside the question of the philosophical plausibility of mathematical fictionalism in its own right, a convincing

[269] For a critical discussion of the idea that fictionalism can explain mathematical truth, see Papineau (1990) pp. 173 and 177-180.

[270] Mendell (2004). In fact, Aristotle's program was more ambitious; unlike fictionalists, he never gave up the notion of objective mathematical truth. This is another reason why his view of mathematics is still worth taking seriously.

[271] Lear (1982) pp.191-192; Hussey (1992) pp. 117-119.

fictionalist interpretation of Aristotle's theory of mathematics must address aforementioned problems, namely: (i) whether such an interpretation is compatible with Aristotle's scientific realism, and (ii) whether fictionalism can provide an account of the truth of mathematics consistent with Aristotle's theory of truth. Although the detail of various fictionalists interpretations of Aristotle's own theory of mathematics vary, all fictionalists endorse the claim that, in Aristotle's philosophy of mathematics, the existence of mathematical objects is not necessary in order to explain mathematical truth.[272] I have argued that such a conception of truth is not Aristotle's, and is not consistent with Aristotle's philosophy.

2. Idealistic Constructivism

The position we will examine in this section is that mathematical objects are constructed by imposing mathematical forms on pure extension obtained from physical objects by abstraction. This interpretation is distinguished from Lear's in that, while Lear argues that "not only the spatial 'matter' of geometrical constructions and theorems is supplied by the sensible, physical realm, but also at least the basic elements of the 'formal' aspect

[272] For example, Hussey argues that mathematical objects for Aristotle are representative objects; and a representative object is true of the class of all individual members which it represents, in virtue of the fact that it possesses just those properties which all the members share (see Hussey (1991) p. 118). Lear also subscribes to this view (see Lear (1982) pp. 190-191). However, this view does not resolve, among others, the precision problem; if there is no perfect triangle in the physical world, for instance, how can a geometrical triangle represent a structure shared among a certain group of physical things? For a general account of representativism, see Fine (1985).

of these constructions and theorems,"[273] on the view of this second position, only the matter of geometrical objects is acquired from sensible objects by abstraction.

The two positions diverge in their interpretations of the statement that mathematical objects exist as matter (τὰ μαθηματικὰ ἔστι ὑλικῶς). As we have seen, Lear interprets the 'matter' in the phrase as meaning the parts of which the whole is composed, namely, the elements of geometrical figures. But this alternative position reads the statement to mean that only the matter of mathematical objects (ἡ ὕλη τῶν μαθηματιῶν) exists in sensible objects. The interpretation is developed by spelling out what the matter of mathematical objects is.

Aristotle certainly accepts the existence of the matter of mathematical objects, since he asks a question which would make no sense if he did not:

> In general one might raise the question, to which science it belongs to discuss difficulties concerning the matter of the objects of mathematics. Not to natural science because the whole business of the natural science is with the things that have in themselves a principle of movement and rest.
>
> ὅλως δ᾽ ἀπορήσειέ τις ἂν ποίας ἐστὶν ἐπιστήμης τὸ διαπορῆσαι περὶ τῆς τῶν μαθηματικῶν ὕλης. οὔτε γὰρ τῆς φυσικῆς, διὰ τὸ περὶ τὰ ἔχοντα ἐν αὑτοῖς ἀρχὴν κινήσεως καὶ στάσεως τὴν τοῦ φυσικοῦ πᾶσαν εἶναι πραγματείαν, οὐδὲ μὴν τῆς σκοπούσης.[274]

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[273] White (1993) p.181. Also, see Mueller (1990) p. 464.

[274] *Met.*, XI, 1, 1059b14-18.

In this passage, he assumes the existence of the matter of mathematical objects, and also argues that the matter does not have in itself a principle of movement and rest. This is consistent with his thesis that mathematical objects are not changeable.[275]

A few reasons have been suggested for the necessity of positing mathematical matter in Aristotle's metaphysics. First of all, a mathematical figure is not a pure form;[276] for Aristotle, the only pure form is God, as defined in *Met*. III, 9.[277] If a figure does not exist as a pure form, there must be the matter in which the form of the figure is embodied.[278] Aristotle says, "There is some matter in everything which is not an essence and a bare form but a 'this.'"[279] Secondly, the matter of geometrical figures is required for individuating the qualitatively identical figures. In the practice of geometry, a geometrical figure is treated as an individual, e.g., a geometer deals with *this* or *that* triangle rather than triangularity itself. And it is common in geometry to treat two or more exactly congruent figures, e.g., two semicircles in a circle; the form of the two semicircles is the

[275] *Met*. III, 2, 997b13-25; 5, 1002a32-b10, etc.

[276] For this point, see Gaukroger (1980) p188; Hussey (1982) p.130.

[277] A part of the human soul, νοῦς, is also matterless. But that does not mean that the whole human soul is independent of sensible matter.

[278] According to Ross, in *Met*., VII, 10, Aristotle lists four kinds of entities as candidates for substance: (i) the pure form, (ii) the intelligible individual, (iii) the material universal, and (iv) the sensible individual. Of these, Ross argues, only (i) does not contain matter; he further identifies (ii) with a mathematical object. See Ross (1924) Vol. I, p.c.

[279] *Met*. VII, 11, 1036b35-37a2. Mueller says, "However, some of the things Aristotle says about geometric objects suggest an analysis in terms of form and matter, and, because of the intuitive similarity between geometric objects and ordinary things, such an analysis has a certain plausibility. For, if ordinary things can be thought of as compounds of theoretically separable but actually inseparable properties and qualityless matter, why shouldn't geometric objects be thought of as compounds of quantitative properties and indefinite extension?" (Mueller (1987) p. 251)

same but the semicircles are not identical with each other, i.e., they are numerically different. The fact that two exactly congruent figures can exist provides the second ground for assuming the matter of mathematical objects. Since their form is exactly the same, if the matter of the figures did not exist, there would be no way to individuate one from the other.[280]

The third reason is related to his thesis of the inseparability of mathematical objects. If there is any thesis Aristotle consistently maintains in his philosophy of mathematics, it is that mathematical objects cannot exist by themselves, i.e., they are not substances. Thus, for Aristotle, "number and shape are always properties, or properties, of something"[281]; and there must be something of which geometrical objects exist as the properties, namely, some substratum of mathematical properties; and that is the matter of geometrical figures.

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[280] See Bostock (1994) pp. 156-157. For Aristotle, when two things share the same form, matter plays the role of individuator. For instance, Aristotle says, "And the whole already completed whole, such and such a form in this flesh and in these bones, is Callias or Socrates; and they are different because of their matter (for that is different), but the same in form (for their form is indivisible) (τὸ δ᾽ ἅπαν ἤδη, τὸ τοιόνδε εἶδος ἐν ταῖσδε ταῖς σαρξὶ καὶ ὀστοῖς, Καλλίας καὶ Σωκράτης: καὶ ἕτερον μὲν διὰ τὴν ὕλην (ἑτέρα γάρ), ταὐτὸ δὲ τῷ εἴδει (ἄτομον γὰρ τὸ εἶδος)) (*Met.*, VII, 8, 1034a5-8)." cf. *Ibid.*, XIV, 2, 1089b15-28.

Many commentators agree that the mathematical matter is necessary for explaining how two qualitatively identical geometrical figures can be differentiated (see Mueller (1987) p. 251; Bostock (1994), pp.156-157; Gaukroger (1982) p.319). However, it is controversial what the principle of individuation for Aristotle is. For the problem of the individuation in Aristotle, see Appendix, n. 437.

[281] Gaukroger (1980) p. 188.

It seems clear that Aristotle identifies the matter of geometrical figures with intelligible matter.[282] He says:

> But matter is unknowable in itself. And some matter is sensible and some intelligible; sensible [matter], for instance bronze and wood and all changeable matter, and intelligible matter, that which is present in sensible things not *qua* sensible, e.g., mathematical objects.
>
> ἡ δ᾽ ὕλη ἄγνωστος καθ᾽ αὑτήν. ὕλη δὲ ἡ μὲν αἰσθητή ἐστιν ἡ δὲ νοητή, αἰσθητὴ μὲν οἷον χαλκὸς καὶ ξύλον καὶ ὅση κινητὴ ὕλη, νοητὴ δὲ ἡ ἐν τοῖς αἰσθητοῖς ὑπάρχουσα μὴ ᾗ αἰσθητά, οἷον τὰ μαθηματικά.[283]

This has to be accounted a clear statement that the matter of the objects of mathematics is intelligible matter.[284]

Concerning mathematical matter, we have so far confirmed two claims as being Aristotle's own on a textual basis: (i) that Aristotle accepts the existence of mathematical matter as distinguished from sensible matter, and (ii) that Aristotle identifies the mathematical matter with intelligible matter (ὕλη νοητή).[285] What is not clear from the text is first, why mathematical matter must be something other than sensible matter, and secondly, what intelligible matter is—in the text it only figures as the matter of geometrical figures.

[282] Gaukroger argues that "the idea of noetic matter is introduced by Aristotle in the context of geometry, and his argument is intended to secure that the geometrical properties that are abstracted from sensible bodies do not thereby become independent (Graukroger (1982) p. 319)."
[283] *Met.*, VII, 10, 1036a8-12.
[284] For the naïve realistic perspective on the intelligible matter, see Lear (1982) p. 181.
[285] Bostock (1994) p. 156.

In response to the first question, an account we can find in the previous two passages (1059b14-18; 1036a8-12) is that, while mathematical objects are unchangeable, sensible matter such as bronze or wood is subject to change; so that sensible matter cannot be the matter of mathematical objects. However, for Aristotle, geometry is the study of a sensible object not *qua* changeable but *qua* limited extension, such as line, figure or solid. Thus, according to my analysis of abstraction, geometry studies only the *kath' hauta* properties of a certain limited extension selected out of the necessary properties of a sensible object. Since the process of abstraction filters out all properties other than those *kath' hauta* properties, it does not matter whether the matter of mathematical figures is changeable or not. Therefore, from the fact that mathematical objects are not changeable, it does not follow that their matter should also be unchangeable. We can recall Aristotle's own example to illustrate the difference between mathematics and physics: While physics studies a nose *qua* snub, geometry studies it *qua* concave. Since the definition of concavity does not include its matter, which is sensible and undergoing change, the concavity itself can be seen as unchnageable. Moreover, if we can regard the concavity in a snub nose as a geometrical object, it seems that at least some geometrical figures are realized in sensible matter.

To explain why the concept of mathematical matter is required apart from sensible matter, we need to take stock once more of the precision problem. The precision problem occurs because some geometrical objects lack instantiations in the physical world. But, as it has been argued, since all geometrical figures should have their matter, those geometrical objects should also have their matter of some kind. Now, since these figures are not actualized in any sensible objects, their matter cannot be sensible. Therefore, some geometrical figures' matter must be something other than sensible matter. And we saw that Aristotle identifies such matter with intelligible matter. In view of his distinction between intelligible and sensible matter, it seems to be a natural consequence that he also distinguishes intelligible figures from sensible and identifies only the former with geometrical figures.[286] Aristotle says:

> But when we come to the concrete thing, e.g., *this* circle, i.e., one of individual circles, whether sensible or intelligible (I mean by intelligible circles the mathematical, and by sensible circles those of bronze and of wood), of these there is no definition, but they are known by thought or perception…
>
> τοῦ δὲ συνόλου ἤδη, οἷον κύκλου τουδὶ καὶ τῶν καθ᾽ ἕκαστά τινος ἢ αἰσθητοῦ ἢ νοητοῦ λέγω δὲ νοητοὺς μὲν οἷον τοὺς μαθηματικούς, αἰσθητοὺς δὲ οἷον τοὺς χαλκοῦς καὶ τοὺς ξυλίνουστούτων δὲ οὐκ ἔστιν ὁρισμός, ἀλλὰ μετὰ νοήσεως ἢ αἰσθήσεως γνωρίζονται...[287]

[286] See Gaukroger (1980) pp.187-188. A similar dichotomy can be found in Aristotle's treatment of numbers, where he distinguishes arithmetical number from sensible numbers in several places. Among them, the following passage may be the most revealing: "Now where it is thought impossible to take away or to add, there the measure is exact. Hence that of number is most exact; for we posit the unit as indivisible at all; *and in all other cases we imitate this sort of measure* (ὅπου μὲν οὖν δοκεῖ μὴ εἶναι ἀφελεῖν ἢ προσθεῖναι, τοῦτο ἀκριβὲς τὸ μέτρον (διὸ τὸ τοῦ ἀριθμοῦ ἀκριβέστατον: τὴν γὰρ μονάδα τιθέασι πάντῃ ἀδιαίρετον: ἐν δὲ τοῖς ἄλλοις μιμοῦνται τὸ τοιοῦτον) (*Met.*, X, 1, 1052b35-1053a2)." cf. *Ibid.*, XIII, 8, 1083b16; XIV, 5, 1092b22-25. I discuss Aristotle's distinction between two different kinds of number in Appendix, 3.

[287] *Ibid.*, VII, 10, 1036a2-6.

Aristotle explicitly here says that mathematical circles are intelligible circles, and their matter is intelligible.[288] Nevertheless, such an identification appears to be in conflict with his assertion that there are perfect physical instantiations of some geometrical objects such as circles or straight lines. Since these instantiations are embodied in sensible matter, they must be sensible objects. One way of squaring these difficulties might be to say that the fact that some geometrical figures have perfect physical instantiations does not imply that their matter must be sensible; it is possible that, while some geometrical figures can be embodied only in intelligible matter, some can be embodied in both intelligible and sensible matter. However, it would seem to be odd that some geometrical figures are actualized in both sensible and intelligible matter, but others only in intelligible matter; if they are ontologically the same kind of entities, we could expect that their matter would be the same.

In seeking to clarify this question of the matter of geometrical figures, let us suppose a physical bronze triangle fully meeting the geometrical definition of a triangle. In this case, clearly the matter of this triangle is sensible. However, from the fact that the bronze triangular shape perfectly satisfies the definition of a triangle, it does not follow that the triangle in bronze is an object of mathematics. Geometry certainly studies triangles which meet the definition of triangles. However, if a geometer can construct more complicated figures such as a figure with 100 sides, there is no need for him to

[288] Notice that, besides his identification of intelligible figures with geometrical figures, Aristotle also maintains that a geometrical figure is an individual, regardless of whether it is perceptible or intelligible (*Ibid*., 11, 1036b33-37a5). I have argued that this is one of the reasons why Aristotle's theory requires the matter of mathematical objects.

obtain triangularity by abstraction from a sensible object. It is merely coincidental that some physical objects perfectly satisfy the definitions of certain geometrical figures. However, insofar as their matter is sensible, they are sensible figures; as Aristotle maintains in the above passage, only intelligible figures qualify as mathematical objects. Thus, the triangles which geometry studies are intelligible triangles whose matter is intelligible, and intelligible triangles can be obtained in the same way that other more complex figures are obtained; namely, they are constructed by the mind's imposing their forms on intelligible matter. Thus, whether or not there are some physical objects which perfectly realize certain geometrical forms, the distinction between mathematical objects and sensible objects according to their matter is maintained.

So much for the first question. Now let us consider the second question: What is intelligible matter? One difficulty in explaining what intelligible matter is is that we do not have enough textual information on the theme; Aristotle mentions intelligible matter only three times throughout his entire corpus, and nowhere gives an explicit account of it. Nevertheless, a few things can be inferred from what has been discussed: First, intelligible matter must be present in sensible things; Aristotle says that it is "present in sensible objects, but not in so far as they are sensible (ἐν τοῖς αἰσθητοῖς ὑπάρχουσα μὴ ᾗ αἰσθητα)."[289] Secondly, intelligible matter is obtained by abstraction from a sensible object; since it is in a sensible object, in order to be the matter of mathematical objects which are not sensible but intelligible, it must be separated from the sensible

[289] See *Met.*, VII, 10, 1036a11.

elements of the sensible object to which it belongs; and we saw that Aristotle's abstraction is the method of such separation.[290]

Commentators generally agree that intelligible matter for Aristotle is pure extension. This interpretation can be traced back to Alexander,[291] and is supported by most modern commentators.[292] Its initial plausibility is that the pure extension has the two characteristics of intelligible matter we just have identified: that it is in a sensible object and can be obtained from a sensible object by abstraction.

In *Categories*, quantity appears as the second category of being which is said of or in a substance. Aristotle divides it into two kinds:

> Thus some quantity is a plurality if it is numerable, and a magnitude if it is measurable. And a plurality is said to be potentially divisible into what is not continuous, while a magnitude [said to be divisible] to what is continuous. Of magnitudes, that which is continuous in one dimension is length; in two dimensions, breadth; in three dimensions, depth. Of these, multiplicity, when it is limited, is a number; length, a line; breadth, a surface; depth, is a body.
>
> πλῆθος μὲν οὖν ποσόν τι ἐὰν ἀριθμητὸν ᾖ, μέγεθος δὲ ἂν μετρητὸν ᾖ. λέγεται δὲ πλῆθος μὲν τὸ διαιρετὸν δυνάμει εἰς μὴ συνεχῆ, μέγεθος δὲ τὸ εἰς συνεχῆ: μεγέθους δὲ τὸ μὲν ἐφ' ἓν συνεχὲς μῆκος τὸ δ' ἐπὶ δύο πλάτος τὸ δ' ἐπὶ τρία βάθος.

[290] Abstraction is a general method by which a science selects a certain group of properties for its investigation; but he uses expressions such as 'τὰ ἐξ ἀφαιρέσεως λεγόμενα' or 'τὰ ἐν ἀφαιρέσει λεγόμενα' only for referring to mathematical objects.

[291] *On Aristotle's Metaphysics,* 510.3; 514.

[292] See Gaukroger (1982) pp. 318-320; White (1993) pp. 179-180; Hussey (1983) pp. 104-5; Lear (1982) pp. 181-182; Mueller (1987) p. 251; Barnes (1985) p. 117; Mueller (1970) pp. 163-167; Gaukroger (1980) pp. 187-189; Hussey (1992) pp. 122 and 130; Bostock (1994) pp. 284-285.

> τούτων δὲ πλῆθος μὲν τὸ πεπερασμένον ἀριθμὸς μῆκος δὲ γραμμὴ πλάτος δὲ ἐπιφάνεια βάθος δὲ σῶμα.[293]

According to this passage, there are two kinds of quantity: that which is continuous and that which is not so. The former is again divided into three kinds: length, breadth, and depth; when these are limited, these quantities become in turn a line, a surface, and a body, respectively. Thus continuous quantity is *in* a substance or predicated of a substance.[294] In *Categories*, only the individual man or horse is called a substance in the strict sense.[295] But, if A can be said to be in B because A is predicated of B, extension in one, two, and three dimensions also can be said to be *in* any sensible thing which is extended. Thus, it can be said that the extension of a sensible object meets the first condition for intelligible matter, that of being in sensibles.[296]

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[293] *Met.* V, 13, 1020a7-14

[294] In fact, Aristotle maintains that every actual thing has magnitude. He says, "Since, as it seems, no actual thing exists in separation apart from sensible magnitudes, the objects of thought are included among the sensible forms; both are those that are spoken of as in abstraction and those which are states and affections of sensible objects (ἐπεὶ δὲ οὐδὲ πρᾶγμα οὐθὲν ἔστι παρὰ τὰ μεγέθη, ὡς δοκεῖ, τὰ αἰσθητὰ κεχωρισμένον, ἐν τοῖς εἴδεσι τοῖς αἰσθητοῖς τὰ νοητά ἐστι, τά τε ἐν ἀφαιρέσει λεγόμενα καὶ ὅσα τῶν αἰσθητῶν ἕξεις καὶ πάθὴ) (*DA*,III, 8, 432a3-6)."

[295] *Cat*., 5, 2a11.

[296] We also find similar accounts of magnitude in *De Anima*. According to Aristotle, we perceive through sense faculties not just the special sensibles of each particular sensible object, such as color and sound, but also common sensibles such as motion, rest, magnitude, number, and figure (see *DA.,* II, 6, 418a18-19; III, 1, 425a14-20). Among these, extension is identified with the continuous quantity, and a figure is sorted out as a species of magnitude (*Ibid.*, III, 1. 425a17-18). Since 'magnitude (μέγεθος)' means spatial magnitude in Aristotle, it can be fairly translated into 'extension.'

Now, let us consider whether pure extension can be separated from a sensible object by abstraction. Most commentators agree that pure extension is acquired by abstraction. They also argue that the abstraction involved in that process takes place in two different stages.[297] For instance, Gaukroger argues that:

> In the case of geometry, Aristotle employs two quite different kinds of abstraction. The first kind involves disregarding the matter of sensible objects so that we are left with properties like being triangular and being round…And whatever, most generally speaking, is round is something that we arrive at by the second kind of abstraction, in which we disregard the properties of sensible objects so that what has these properties becomes the object of investigation. What we are left with is a substratum of indeterminate extension characterized solely in terms of its spatial dimensions: length, breadth and depth. Such a substratum cannot be sensible since it has been deprived of the properties that would render it sensible; nor can it be something which is independent since it is simply an abstraction. It is this substratum that Aristotle calls intelligible matter.[298]

I agree generally with the basic idea that length, breadth and depth can be obtained from a sensible object by abstraction. Aristotle's abstraction allows to us separate any true predicate from the other predicates of the subject. Thus, we can obtain 'having length'

[297] See Gaukroger (1982) p. 139 and (1980) p. 188; Hussey (1983) pp. 104-5; Ross (1924) Vol. I, p. liv. Most commentators agree that intelligible matter can be abstracted from sensible objects, and that it may be understood as extension without sensible properties. However, accounts of the nature of extension differ considerably in their detail. For instance, Gaukroger identifies it with length, breadth and depth (Gaukroger (1980) p. 188); Hussey identifies it with pure extension which does not have any property whatsoever like prime matter (Hussey (1992) pp. 122 and 130), and White assumes it to be a line, a surface, and a body (White (1993) pp. 179-180). Some commentators identify prime matter with extension (see Sokolowski (1970) pp. 263-288; Sorabji (1986) pp. 1-22).

[298] Gaukroger (1980) p. 188.

from any sensible object, so long as it extends in at least one dimension. However, Gaukroger's description of the whole process of the mind's acquiring pure extension of each dimension by abstraction remains unclear. In the passage quoted above, by 'abstraction', Gaukroger describes retaining the property in question, extension, and disregarding all other irrelevant properties. But since he does not provide his own account of Aristotelian abstraction, it is not easy to see exactly what he means by 'disregarding.' There are two possible readings. (i) If we consider abstraction as a linguistic analysis which is a part of linguistic functional calculation selecting from all of the concepts which are true of the sensible object those which are intrinsically true of it only under a certain description, then, 'disregarding' could refer to the conceptual operation of 'filtering out'[299] the concepts which only happen to be true of the sensible object under that description. Alternatively, (ii) if abstraction is understood in the more traditional way as the process of acquiring a universal from particulars, 'disregarding' would involve a mental attitude toward a sensible thing such as ignoring or paying no attention to a certain aspect of it.[300]

[299] Lear (1982) p. 168.

[300] According to the traditional interpretation, what we deal with during abstraction is not a thing itself outside the mind but a kind of representation of a thing, a φάντασμα. So it would be more accurate to say that disregarding is a mental act performed on a φάντασμα; namely, ignoring or paying no attention to a certain aspect of a φάντασμα. Avicenna and Thomas both argue that an object's intellectual form is separated from the φάντασμα by abstraction. φάντασμα is the end product of perception, which is involved in both perception and thinking. For the role of φάντασμα in perception, see Nussbaum (1978) and Schofield (1978). Aristotle maintains that thinking is not possible without φάντασμα (*DM.,* 450a 1; *DA.,* 431a 15-20; 432a 8-12).

Reading (i), however, does not require there to be two different kinds of abstraction in order to obtain the concept of extension in each dimension from a sensible object. In my analysis of Aristotle's abstraction, we can acquire the concept of length from any sensible object which is extended in one dimension, i.e., from any sensible line. Recall that, in my view, A studies B *qua* C iff A investigates only *kath' hauta* properties of C which also belong to B; abstraction determines which properties are *kath' hauta* properties by removing all other irrelevant properties. Since A is a *kath' hauto* property of B iff (i) A is a property of B and A appears in the definition of B, or (ii) A is a property of B and B appears in the definition of A, all of length, breadth, and depth are *kath' hauta* properties of extension or the continuous on the grounds that their definitions refer to extension; for instance, length is defined as extension in one dimension, and 'extension' is a part of the definition of length insofar as the definition of length contains the concept of 'extension.' Since length is a *kath'hauto* property of extension, for any physical line, *a*, we can conceive of length only by conceiving of *a qua* extension. Likewise, a geometer can also obtain the concepts of breadth and depth from a sensible object by abstraction. During the process of abstraction, all other irrelevant properties such as weight, hardness, shape, texture, temperature, etc., are removed, leaving only *kath' hauta* properties of the continuous. But this process of separating the concept of extension in each dimension from the other concepts does not require two different kinds of abstraction; abstraction is needed only once during the whole process.

Since abstraction as a conceptual separation is required only once for acquiring the concept of extension from a sensible object, it is more likely that Gaukroger is interpreting abstraction in the traditional way and using the term ‘disregarding’ in the sense of the reading (ii). However, if ‘disregarding’ can be read as a kind of mental act such as ‘ignoring’ or ‘paying no attention,’ it is difficult to see how extension in each dimension can be obtained from a sensible object by ‘disregarding’ all other properties in order. Suppose that there is a bronze circle, and, as the passage explains, attempt to ‘disregard’ the matter of the bronze sphere, namely, all the properties of the matter such as ‘being brazen,’ ‘being brown,’ ‘being hard,’ etc., and then ‘disregard’ its shape, namely, being circular. What remains after disregarding all these properties? According to Gaukroger, as the result of this ‘disregarding,’ we should be left with a substratum of indeterminate extension in one dimension, namely, breadth. However, it seems to me that, if every sensible property such as shape and color is disregarded from a sensible object, very simply nothing will be left in its place. We always perceive extension in each dimension along with other sensible properties. For example, to perceive breadth is always to perceive a colored or shaped expanse, e.g., a red or triangular expanse; we never perceive breadth in itself, without color or some sensible property.[301]

Aristotle divides objects of perception into three kinds: special, common and incidental objects.[302] While there is a special sense organ and sense for special objects, there is no such sense organ or sense for common objects,[303] which are perceived by

[301] Cf. Gaukroger (1982) p. 320.
[302] *DA*., II, 6, 418a8-11.
[303] Aristotle denies the existence of a special sense organ and sense for common objects (*DA*, II, 6). He also, however, argues that there is a common sense for them (*DA*, III, 1,425a27). This seems

more than one sense; a special object is meanwhile perceived by its special sense. Incidental objects are distinguished from the first two in that they do not directly affect any sense faculty. For instance, when the son of Diares is perceived, sight perceives only whiteness and similarly other senses perceive certain other particular qualities;[304] the son of Diares *as such* does not affect any specific sense. We see a white thing which happens to be the son of Diares and in this sense *he* is perceived only incidentally (κατὰ συμβεβηκός).[305]

Aristotle identifies spatial magnitude or extension with one of common objects. Since extension is perceived in common by touch or sight, if there is no special object of sight or touch, it cannot be perceived at all.[306] That is, we cannot perceive extension alone separately from all other sensible objects. Thus, if we disregard all the sensible properties of objects, we will be left without any sense data of extension.

If extension cannot be perceived, it follows it cannot be known. I argued that Aristotle uses the term 'thinking' in two different ways.[307] In the narrow technical sense, thinking of *a* means receiving the (intelligible) form of *a*; and thinking in this sense is

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contradictory. Hamlyn argues that the two claims are compatible, understanding that, for Aristotle, every special sense organ is an organ of common sense as well as of its special sense. Since every sense organ has this common sensing power, it can be said that there is no special sense for common objects, while every sense organ remains receptive to objects in their common sense. See Hamlyn (1993) p.117.

304 *DA*., II, 6, 418a17-22.

305 For a discussion of the incidental objects of perception, see Hamlyn (1993) pp. 105-107 and Modrak (2000) pp. 223-224.

306 For Aristotle's discussion of common objects and common sense, see *DA*, II, 6, 418a7-20; III, 1, 425a14-2, 425b12.

307 See Chapter Two, §6. I will discuss two senses of 'thinking' again more in detail in Chapter Four, §5.

identified with knowing. But, for Aristotle, all objects of thinking (in this narrow sense) are included in the forms of perception; so that thinking of an object, *a,* presupposes the perception of *a*.[308] Thus, if extension cannot be perceived, it seems it cannot be thought, either. But, since extension is perceptible only along with the perception of other sensible objects, if we disregard all the sensible properties of a sensible object, we will be unable to perceive its extension. Since thinking in this sense means knowing, without a perception of extension, there will be no knowledge of extension either.

So far, we have seen what intelligible matter is and how it is acquired from sensible objects. But matter which is itself indeterminate in each dimension cannot be

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[308] Aristotle says, "Since, as it seems, no actual thing exists in separation apart from sensible magnitudes, the objects of thought are included among the sensible forms; both are those that are spoken of as in abstraction and those which are states and affections of sensible objects. And for this reason, without perceiving, one would not learn or understand anything, and when one contemplates it is necessary to contemplate some *phantasma* at the same time; for *phantasmata* are like sense-perceptions, except in that they are without matter (ἐπεὶ δὲ οὐδὲ πρᾶγμα οὐθὲν ἔστι παρὰ τὰ μεγέθη, ὡς δοκεῖ, τὰ αἰσθητὰ κεχωρισμένον, ἐν τοῖς εἴδεσι τοῖς αἰσθητοῖς τὰ νοητά ἐστι, τά τε ἐν ἀφαιρέσει λεγόμενα καὶ ὅσα τῶν αἰσθητῶν ἕξεις καὶ πάθη. καὶ διὰ τοῦτο οὔτε μὴ αἰσθανόμενος μηθὲν οὐθὲν ἂν μάθοι οὐδὲ ξυνείη, ὅταν τε θεωρῇ, ἀνάγκη ἅμα φάντασμά τι θεωρεῖν: τὰ γὰρ φαντάσματα ὥσπερ αἰσθήματά ἐστι, πλὴν ἄνευ ὕλης) (*DA*, III, 8, 432a3- 10) ."

For Aristotle, a *phantasma* is the end product of perception and provides us with all sensible information concerning the perceived object; namely, it is the sensible representation of a sensible object. Since the objects of thought, i.e., intelligible forms, are included in the forms of perception, to think is to grasp an intelligible form in *phantasma*; in that sense, *phantasma* is also regarded as the material of thinking by Aristotle. And, since the objects of thinking are included in the forms of perception, Aristotle argues that we cannot think of anything without its *phantasma*.

Traditionally, abstraction was understood as the separation of an intelligible form from *phantasma* by ignoring the form's irrelevant aspects, i.e., its sensible properties. However, as I have argued so far, if we take away every sensible property from *phantasma*, we will be left with nothing.

the object of geometry; geometry deals with extension that is limited, namely, with particular lines, figures, and solids. Mueller argues that such geometrical objects are the results of the intellect's imposing geometrical forms on abstracted extension:

> First, there are the basic objects: points, lines, planes, solids. The last three are conceived of as indeterminate extension and, therefore, as matter on which geometric properties are imposed. The imposition of these properties produces the ordinary geometric figures, straight or curved lines, triangles, cubes, etc. The definition of such a figure will include both the form, the properties imposed, and the matter; but in the definition this matter will also play the role of genus. A circle is a plane figure.[309]

But, according to the second position, the mathematical forms which the intellect imposes on intelligible matter are not in sensible objects. At the same time, since Aristotle denies their independent existence, they cannot exist apart from sensible objects, either. If the forms do not exist either in sensibles or apart from sensibles, where does our mind obtain them from? At this point, we should recall once more that for Aristotle the geometrical constructions (διαγράμματα) in geometrical proofs are constructed by thought. Aristotle says nothing explicit about the mental construction of geometrical objects. But, if we accept that geometrical figures can be constructed in the same way as διαγράμματα, as I pointed out, it should be also possible to construct geometrical

[309] Mueller (1970) p.167.

figures.[310] Further, since διαγράμματα are constructed by thought, geometrical objects should be constructed by thought as well.[311]

One advantage of this position is that it avoids the precision problem. If it is our mind which imposes a geometrical form on intelligible matter, the precision problem never occurs; we know the exact definition of each mathematical object.[312] Since geometrical properties or forms are not obtained from sensible objects, it does not matter whether or not sensible objects perfectly instantiate geometrical properties.

Proponents of the second position would also argue that this interpretation differentiates Aristotle from modern subjective constructivists. Insofar as the concept of extension is acquired from external objects, mathematics is not purely subjective or *a priori*. Since the matter of mathematicals is extension, Aristotle can justly say, in one sense, that mathematical objects exist as matter in sensible objects. But they are not in sensibles because their forms are not in sensibles. Nevertheless, mathematics is a study of sensibles, in the sense that it investigates what properties the extension of sensibles *can* have. In this sense, it might be said that the position not only solves the precision

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[310] As Ackrill points out, the construction of diagrams in Euclidean geometry plays an important role in proving theorems, e.g., *Elements,* I, 47; and many 'propositions' in *Elements* are solutions to construction problems, e.g., *Elements*, I, 1, 2, and 3. See Ackrill (1963) p.111.

[311] The mind's constitutive function in the practice of mathematics can be found elsewhere, too, in particular with respect to arithmetic. In my view, for Aristotle, numbers are fictional entities to an extent that they can be described as mind-dependent. For the fictionality of numbers, see Appendix, 3.

[312] See Mueller (1970) p. 168.

problem, but also is distinguished from idealism.[313] Because of such advantages, this view is most popular among modern commentators of Aristotle.[314]

However, we should be aware that this interpretation obtains all these explanatory advantages only at the expense of sacrificing Aristotle's mathematical naïve realism. Such an incompatibility with naïve realism, however, might well be permissible. We saw that there are several passages suggesting that Aristotle is himself apprised of the problems of mathematical naïve realism. Indeed, some of his claims about mathematical objects are not compatible with naïve mathematical realism. Thus, as entertained in Chapter 3, it is possible that Aristotle developed two incompatible views of mathematics one after another. If this is the case, no interpretation will be able to accommodate the opposite sides of his view.[315] In that sense, the internal inconsistency problem might be passed over in favor of evaluating different interpretations of Aristotle's view; it will be understandable that some commentators do not take naïve

[313] Commentators argue that Aristotle's view preserves a link between mathematics and the sensible world, relying on the fact that the matter of mathematical objects is abstracted from sensible objects. For example, Mueller argues, "Aristotle's account of geometric objects would seem, then, to be something like the following. In his reasoning the geometer deals directly with the particular geometric objects which I have been describing. These objects, though not real in the sense in which sensible substances are, are intimately connected with sensible reality and in a certain sense underlie it (Mueller (1970) p. 171)." Similarly, Hussey says, "but, as with mathematics, there is a matterlike residue of structure which, though exemplified only in the actual world, cannot be wholly reduced to it either ontologically or epistemologically (Hussey, p. 133)."

[314] Some of those who endorse this interpretation are: Hussey (1992); Gaukroger (1980) pp. 187-197; Mueller (1970); White (1993); Gaukroger (1982) pp. 312-322.

[315] But notice that my analysis of abstraction is compatible with both the naïve realistic interpretation and the second position; in my account, it is also possible to obtain extension in each dimension by abstraction. Thus, properly understood, Aristotle's theory of abstraction is applicable to both his earlier mathematical naïve realism and later fictionalistic theory.

mathematical realism into consideration in characterizing Aristotle's philosophy of number. In this regard, the fact that the second position cannot accommodate the naïve realistic aspect of Aristotle's theory of mathematics is no reason to renounce it.

The real issue is whether a view describing geometrical objects as mental constructions can be made compatible with Aristotle's *scientific* realism. Although the matter of geometrical objects is obtained from the actual world, mathematical objects are still something constructed by the intellect, insofar as their forms are so constructed. Thus, a geometrical object as the composite of intelligible matter and the form remains a fictional entity. Since the form of a geometrical object is essentially constructed, the definition of a geometrical object is not a description of a real object; rather, it is a meaning-postulation for a term. For instance, the definition of triangle only determines the meaning of the term 'triangle'; there is no such thing as something triangular in the geometrical sense. But, if geometry is to be an Aristotelian science, a definition of a geometrical object must be the description of the essence of a real being.

One could respond here that the fact that mathematical forms are created by our mind does not make geometry mind-dependent; the concept of extension constrains the way we define geometrical objects and formulate geometrical axioms. If we had a different concept of extension, that is, if we received a different concept from sensibles, we would have a different geometry. For instance, some theorems in Euclidean geometry hold true only in Euclidean space. [316] On this topic, White says:

[316] See Heath (1956) Vol. I, p. 199.

The question that is not answered in Aristotle's texts, so far as I can see, is to what degree these material, potential data, supplied by the sensible/physical realm, constrain the constructions and theorems that may be actually obtained by the νόησις of geometer… As Hussey aptly notes, 'there is a matterlike residue of [mathematical] structure which, though exemplified only in the actual world, cannot be wholly reduced to it either ontologically or epistemologically' (Hussey, p. 132). But unanswered questions remain as to how and in what degree this 'matterlike residue of structure' constrains 'theoretical' mathematics. From the perspective of mathematical hindsight, we may be inclined to think of this constraint as substantial. For example, the fact that space or extension may be assumed to be Euclidean or 'flat' (as opposed to elliptical or hyperbolic) constrains the following fact: it is triangles that are *Euclidean*, in the sense of satisfying the theorem that the sum of their interior angles is equal the sum of two right angles, that are constructible 'in' that space/extension by geometers. And Sir Thomas Heath notes that Euclid's postulates, such as the postulate licensing the construction of a circle of any circle and any radius that we previously mentioned, may be regarded as 'helping to complete delineation of the Space which Euclid's geometry is to investigate formally.'[317]

But this response is not without its own difficulties. First, as White himself recognizes, it is an anachronism to attribute to Aristotle the idea that our concept of space constrains geometry.[318] Although the debate on Euclid's Fifth (or parallel) Postulate, which led to

[317] White (1993) p. 180

[318] See White (1993) pp. 180-181. Aristotle argues that there are only three kinds of limited magnitude: a line, a surface, and a body: "A magnitude [divisible] in one way is a line, two ways a surface, and three ways a body. And beyond these there is no other magnitude, because the three [dimensions] are all [dimensions], and 'in three ways' means 'all ways.' For, as Pythagoreans say, the whole [universe] and all things are is determined by the number three, since end and middle and beginning give the number of the whole and their number is the triad (Μεγέθους δὲ τὸ μὲν ἐφ' ἓν γραμμή, τὸ δ' ἐπὶ δύο ἐπίπεδον, τὸ δ' ἐπὶ τρία σῶμα: καὶ παρὰ ταῦτα οὐκ ἔστιν ἄλλο μέγεθος διὰ τὸ τὰ τρία πάντα εἶναι καὶ τὸ τρὶς πάντῃ. Καθάπερ γάρ φασι καὶ οἱ Πυθαγόρειοι, τὸ πᾶν καὶ τὰ πάντα τοῖς τρισὶν ὥρισται: τελευτὴ γὰρ καὶ μέσον καὶ ἀρχὴ τὸν ἀριθμὸν ἔχει τὸν τοῦ παντός, ταῦτα δὲ τὸν τῆς τριάδος) (*DC*, I, 1, 268a7-13)." Since geometry also deals with these three

the discovery of non-Euclidean geometries, is as old as Euclid's *Elements*, it is only in the 19th century that the first non-Euclidean geometry emerges. Secondly, in order that the concept of extension abstracted from the sensible world constrain Euclidean geometry, that extension should be Euclidean space, that is, it should be perfectly flat. It is questionable, however, how the concept of flat space can be acquired from a sensible object by abstraction. We saw that a concept or predicate, P, can be obtained from an object, *a*, by abstraction only if it is true that *a* is P; this was the reason why abstraction could not provide a solution for the precision problem—some concepts are true of a geometrical figure but are not predicated of any sensible object. While Aristotle talks about sensible objects which are perfectly circular or straight, he never mentions a perfectly flat object in any passage,[319] and it is also a physical fact that there is no sensible thing which is perfectly even.[320]

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dimensions of space alone, this might be taken as textual evidence for Aristotle's belief that the concept of space constrains geometry in a certain way. But the reason why Aristotle believed that there were only three kinds of magnitudes is irrelevant to the issue we are discussing. It is highly questionable whether he considered how spatial extension could play any predeterminative role in the formation of our geometrical concepts. For the three dimensionality of magnitude, also see *Ibid.*, 7. 274b19-20.

[319] Gaukroger argues on the basis of *DA,* III, 6, 430b20 that, for Aristotle, all spatial geometrical magnitudes, including two-dimensional planes, are generated from lines and points (see Gaukroger (1980) p.189). If he is correct, Aristotle would believe that a flat surface is constructed out of lines and points, rather than being abstracted from the sensible world; this would give Aristotle one way of getting round the precision problem. But more generally Aristotle thinks that points are posterior in being to lines, lines to planes, and planes to solids. Nevertheless, that does not mean that Euclidean flat space does not need to be constructed from such elements as points, or lines; two-dimensionality which can be obtained from any plane by abstraction is not the same as Euclidean flat space; and further not every extension in two dimensions is perfectly even.

[320] The idea of abstracting pure extension from a sensible object is not Aristotle's own. The closest idea is be found in the following passage: "Just as the mathematician makes a study of

In fact, with regard to the issue of compatibility with scientific realism, the second position is no better than Lear's view; it is exposed to all the criticism confronting Lear's constructivist interpretation. We saw that, from the fact that a geometrical figure is composed of elements which are obtained from sensible objects, it does not follow that

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things resulting from abstraction, for after having removed all sensibles such as weight and lightness, hardness and its opposite, heat and cold, and the other sensible opposites, he leaves only the quantitative and continuous, sometimes in one, sometimes in two, sometimes in three dimensions, and the properties of them *qua* quantitative and continuous, and does not consider them in any other respect, and examines their relative positions to others and things which belong to these things, and commensurability and incommensurability of others, and the ratios of others; but nevertheless we hold that there is one and the same science of all these things, i.e., geometry (καθάπερ δ᾽ ὁ μαθηματικὸς περὶ τὰ ἐξ ἀφαιρέσεως τὴν θεωρίαν ποιεῖται ʿπεριελὼν γὰρ πάντα τὰ αἰσθητὰ θεωρεῖ, οἷον βάρος καὶ κουφότητα καὶ σκληρότητα καὶ τοὐναντίον, ἔτι δὲ καὶ θερμότητα καὶ ψυχρότητα καὶ τὰς ἄλλας αἰσθητὰς ἐναντιώσεις, μόνον δὲ καταλείπει τὸ ποσὸν καὶ συνεχές, τῶν μὲν ἐφ᾽ ἓν τῶν δ᾽ ἐπὶ δύο τῶν δ᾽ ἐπὶ τρία, καὶ τὰ πάθη τὰ τούτων ᾗ ποσά ἐστι καὶ συνεχῆ, καὶ οὐ καθ᾽ ἕτερόν τι θεωρεῖ, καὶ τῶν μὲν τὰς πρὸς ἄλληλα θέσεις σκοπεῖ καὶ τὰ ταύταις ὑπάρχοντα, τῶν δὲ τὰς συμμετρίς καὶ ἀσυμμετρίς, τῶν δὲ τοὺς λόγος, ἀλλ᾽ ὅμως μίαν πάντων καὶ τὴν αὐτὴν τίθεμεν ἐπιστήμην τὴν γεωμετρικήν) (*Met*.., XI, 3, 1061a28-35)."

This passage might be regarded as describing the separation of pure extension from sensibles by abstraction. But, in fact, the passage rather supports a naïve realistic interpretation of Aristotle. The phrase 'τὸ ποσὸν καὶ συνεχές, τῶν μὲν ἐφ᾽ ἓν τῶν δ᾽ ἐπὶ δύο τῶν δ᾽ ἐπὶ τρία,' may refer to either 'length, breadth, and depth' or 'line, figure, and body.' But the fact that they have position, and some properties such as commensurability and incommensurability sustaining inter-relationships with each other, indicates that Aristotle is not talking about pure extension but rather particular limited extension. Further, not only 'τὸ ποσὸν καὶ συνεχές, τῶν μὲν ἐφ᾽ ἓν τῶν δ᾽ ἐπὶ δύο τῶν δ᾽ ἐπὶ τρία,' but also 'τὰ πάθη τὰ τούτων ᾗ ποσά ἐστι καὶ συνεχῆ' are left over as the result of abstraction. According to my analysis of '*qua,*' 'τὰ πάθη τὰ τούτων ᾗ ποσά ἐστι καὶ συνεχῆ' signifies the *kath' hauta* properties of extension, which also belong to these particular limited extensions. For instance, if geometry studies a ruler *qua* extension, geometry removes all sensible properties from that ruler to leave only the line of the ruler, proceeding to investigate the *kath' hauta* properties of extension among the properties of the line, i.e., its measurability or ratio to other lines, etc.

it is more real than an imaginary entity, e.g., a triangle is as fictional as a pink elephant is. Likewise, the fact that the matter of mathematical objects is acquired by abstraction from sensible objects does not make them more real than other fictional things. If a geometrical figure is real or has a link to the sensible world due to the fact that its matter, i.e., two-dimensional extension, is acquired from the sensible world, any two-dimensional picture should be as real as a geometrical figure; the matter of the picture and that of the geometrical figure would be the same. In the same vein, there would be no difference between a dragon in a tale and a geometrical cone in terms of reality; the matter of both is three-dimensional extension acquired from the sensible world.

3. Mathematical Objects as Potential Beings

In the previous two sections, we examined two fictionalistic interpretations of Aristotle's view of mathematics. Although both positions resolved the precision problem, they did so at the cost of violating Aristotle's scientific realism: For Aristotle, if mathematical entities are fictional, mathematics is not a science at all. Further, if mathematical statements are not descriptions of real entities, there can be no mathematical truth, given that for Aristotle, truth is a correspondence between what is said (τὸ λέγειν) and what is (τὸ ὂν). Thus, as these sections have established, in order for any interpretation of Aristotle's philosophy of mathematics to be consistent with his own view of science and truth, it must show that mathematical objects can be thought of as *being*, as having some kind of existence in his ontology.

It remains difficult, though, to identify mathematical objects with something existent in these ontological terms. Aristotle claims, on the one hand, that mathematical objects are not separated from sensible objects; but on the other hand, also claims that mathematical objects are not *in* sensible objects. The first claim seems to suggest that mathematical objects either are sensible substances or properties thereof. In denying the substancehood of mathematical objects, though, Aristotle's anti-mathematical Platonism only leaves a possible status for mathematical objects as properties of sensible objects. However, the second claim would also seem to rule out this possibility. If mathematical objects are neither sensible substances nor the properties of these substances, without yet being separated from sensible objects, mathematical objects would seem to have no place in the actual sensible world at all.

3.1. The Matter of Mathematical Objects as Potentiality

Nervertheless, Aristotle does leave room for mathematical objects in his ontology in his remark that there are many senses of being. Aristotle generally parses his frequent comment that being has many senses in terms of four main meanings: (i) 'being' signifies one of the ten ontological categories, such as substance, quality, quantity, etc,[321] (ii) being has two different modes: potentiality and actuality or being potentially and being

[321] *Cat.*, 4; *DA.*, I, 5, 410a13-15; *Met.*, VII, 1, 1028a10-20.

actually,[322] (iii) 'being' means being true,[323] and (iv) being means being accidentally or accidental being.[324] However, as far as the ontological question goes of what kind of entities mathematical objects can be in Aristotle's ontology, the meanings of being at (iii) and (iv) can be excluded; for Aristotle, being as being true is in the province of logic; and an accidental being would not fall under the purview of a science.[325] This leaves the senses of 'being' (i) and (ii) as the sole candidates for answering how Aristotle classes mathematical objects. Since such objects are neither actual sensible entities, nor entities apart from sensible objects, it seems plausible to class them as a kind of potential being, regardless of whether they are substances or properties of substances. Such a demonstration—showing that mathematical objects are a sort of potential being—seems the only way to avoid the issue of incompatibility between Aristotle's philosophy of mathematics and his scientific realism.

Although the idea that mathematical beings are potential beings is not explicit in the text, certain passages lend it textual warrant. One such is the passage that forms the starting-point of all the fictionalistic interpretations. Confirming that mathematical objects exist, Aristotle says:

> Thus, then, geometers speak correctly, and they talk about existing things, and they do exist; for being has two forms—it exists not only in actuality but also as matter.

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322 *Phys*., III, 6., 206a1 4-15; *Met.*, IX, 1, 1045ba32-35; VI, 2, 1026b2.

323 *Met*., VI, 2, 1026a34-35.

324 *Ibid*., 1026b1-2; Ross (1924) Vol. I, p. xvii.

325 *Ibid*., 1026b3-4.

> ὥστε διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν: διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ᾽ ὑλικῶς.[326]

Aristotle's intention would seem clear: Although mathematical objects do not exist in actuality, they can be said to exist on the strength of the existence of another way of being, being as matter, i.e., mathematical objects exist as matter. However, as suggested in the previous chapter, it is not clear what Aristotle means by 'matter' in the passage. We examined two different views: Lear interprets 'matter' as basic geometrical elements such as straight lines, circles, etc., whereas many other commentators have understood it as intelligible matter, pure extension. Both these views, however, share the problematic feature that their interpretations of 'matter' end up making geometrical objects fictional entities. It may also be possible, though, to interpret 'matter' as 'potentiality' or 'potential being'.[327] Two considerations support this assimilation of matter to potentiality. First, in the above passage, Aristotle contrasts the term 'matter' with actuality, invoking 'being matter' as another form of being. Typically, Aristotle's discussion of different senses of being pairs 'being in potentiality' and 'being in actuality', making this passage exceptional insofar as Aristotle would seem to frame an explicit contrast between matter and actuality. Secondly, Aristotle frequently uses the two terms 'matter' and 'potentiality'

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[326] *Ibid*., XIII, 3, 1078a28-31.

[327] Many commentators accept this interpretation. See Mignucci, p. 183; White (1993), pp. 168 and 178; Ross (1924) Vol. I, p. cvi; Vol. II, p. 418; Kirwan (1993) pp. 225-6; Hamlyn (1993) p. 84; Hintikka (1996) p. 207.

interchangeably.[328] For Aristotle, every change is the actualization of something potential, or a transition from something potential to something actual, e.g., a transition from being visible to seeing, or from a baby—a potential man—to a man. In this sense, it can be said that potentiality is the starting point of a change and actuality its end state. The relationship between a substance (or substance-like thing like an artifact) and its matter can also be understood in terms of the actualization of potentiality, in that "it is because some matter is a suitable starting point for the production of something, by the exercise of a capacity, that matter stands to that result as the potential to actual."[329] In this respect, the matter of geometrical object might be taken to constitute a kind of potentiality in serving as the starting point for constructing a geometrical figure.

3.2. Aristotle's Homonymous Uses of 'Potentiality'

Aristotle's ontology is more generous than that of modern physicalists, in that it regards not only actual beings but also potential beings as real entities. So in order to show that mathematical objects are real entities for Aristotle, it should be sufficient to show that they are beings in potentiality. Given that mathematical objects exist as matter, and that matter sometimes means potentiality for Aristotle, the existence of mathematical objects seems to have been shown.

[328] *Phys*, III, 6, 206a14; 206b13-16; *DA*., II, 1, 412a6-11; *Met*., III, 5, 1002a20-25; VIII, 2, 1042b8-11; 1043a14-17; IX, 6, 1045a30-35; 7, 1049a19-24; 8, 1050a15-16; XII, 4, 1070b10-13; XIII, 3, 1078a28-31; XIV, 1, 1088b1-2, etc.

[329] Makin (2006) p. 158. For Aristotle's mapping of matter-substance onto potentiality-actuality, see *Met*., IX, 7, 1048b35-1049b1.

One problem with this approach lies with Aristotle's tendency to use the term 'potentiality' homonymously. If 'potentiality' has various senses, we first need to check the sense in which Aristotle takes it as a mode of existence other than actual existence. Next, we need to examine whether this is the sense of potentiality designated as being coextensive with matter. Only when potentiality in the sense of matter is confirmed as something existent in Aristotle's ontology, can we agree that the fact of mathematical objects' existing as matter entails the existence of mathematical objects.

We can begin by considering the primary sense of potentiality, an important task in the light of Aristotle's assertion that all potentialities are so called in reference to one primary kind.[330] Aristotle defines this primary potentiality as "a principle of change in something else or in the thing itself *qua* something else (ἡ ἀρχὴ κινήσεως ἢ μεταβολῆς ἡ ἐν ἑτέρῳ ἢ ᾗ ἕτερον)."[331] According to this definition, the notion of potentiality is introduced to provide an account of how change in general is possible.[332]

Aristotle argues that the introduction of a concept of potentiality is necessary to avoid the difficulties involved in Megarian actualism. According to Aristotle's report, the Megarian school maintains that 'a thing can act only when it is acting, and when it is not acting it cannot act';[333] for instance, a builder can build only when he is building. Commentators label this thesis actualism in that it accepts only what is actual as being

[330] *Met.*, IX, 1, 1046a9-16
[331] *Ibid.*, V, 12, 1019a15-20; 1019b34-1020a7; IX, 1, 1046a9-16.
[332] *Ibid.*, IX, 2, 1046b29-47a30
[333] *Met.*, IX, 3, 1046b29-30.

existent.[334] For Aristotle, this actualism is fraught with difficulties. Centrally, it cannot explain how a thing is changed: If something can happen only when it is happening, what is not happening cannot happen; hence it never will happen.[335] Then, Aristotle puts it, "that which stands always stands, and that which sits will always sit since if it is sitting it will not get up (ἀεὶ γὰρ τό τε ἑστηκὸς ἑστήξεται καὶ τὸ καθήμενον καθεδεῖται: οὐ γὰρ ἀναστήσεται ἂν καθέζηται: ἀδύνατον γὰρ ἔσται ἀναστῆναι ὅ γε μὴ δύναται ἀναστῆναι)."[336] To avoid this absurdity, Aristotle argues, it is necessary to accept the existence of a certain inactive power or capacity (δύναμις) which stands as a cause of any changes; it is inactive in the sense that it remains inactive until it is actualized.[337] For instance, a builder can build even if he is not building on the basis of his capacity to build; and water can be boiled on the basis of its having a capacity of being boiled in certain circumstances. Because it is inactive until actualized, this power or capacity is also contrasted with actuality and translated as 'potentiality'.[338] Potentiality in this sense is understood as a causal power or a dispositional property standing in contrast with an active property; it is exercised "unless something prevents and hinders it (ἂν μή τι κωλύῃ)".[339]

[334] Witt (2003) p. 15
[335] For Aristotle's report and criticism on the Megarian school's actualism, see *Met.,* IX, 3, 1046b29-10447a30.
[336] *Ibid*., 1047a15-16
[337] *Ibid*., 1047a20-30
[338] In the ordinary usage of Greek 'δύναμις' means power and also ability or capacity. See Witt (2003) p.9.
[339] *Phys*., VIII, 4, 255b11; 225b24; *Met*., IX, 7, 1049a8-10

Previously, mathematical matter was interpreted either as the basic elements of geometrical figures or as pure extension. While my reading of Aristotle's text comes down on the side of the latter interpretation, mathematical matter for either interpretation hardly meets a description of potentiality in its primary sense. When we say that the matter of geometrical figures, i.e., their pure extension, *is in* a sensible object, this cannot mean that it exists in the object as a causal power or a dispositional property towards the realization of certain geometrical figures in the actual world.

3.3. Potentiality as Another Mode of Existence

Nevertheless, Aristotle does seem to leave available another important sense of δύναμις not reduced to its primary meaning.[340] Although Aristotle's prime example of the relation of potentiality and actuality is that of a capacity and its exercise, e.g., sight and seeing, he also applies the distinction to other kinds of beings as well. For instance, Aristotle understands a boy as a potential human being and a man as a complete human being. That is, for Aristotle, not only inactive causal power but also an incomplete substance is an example of a potential being.[341]

Commentators draw different implications from the fact that this division of potentiality and actuality is applied cross-categorically, generally dividing into two main approaches: unitarian and dualistic. The unitary interpretation denies that there is a new

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[340] *Met.*, IX, 1; XII, 5, 1071a3-17; XII, 5, 1046a10-11.

[341] *Met.*, IX, 5, 1048a37-b6.

meaning of δύναμις at issue here.[342] According to this interpretation, in describing the actualization of substance, Aristotle simply introduces a different kind of causal power. For instance, Ross finds a special kind of causal power governing the internal development of substances in Aristotle.[343] By contrast, dualists argue that 'δύναμις' cannot name a certain type of causal power. On this view, Aristotle, especially in *Met.,* IX, 6, 1048a37-1048b6, distinguishes two ways of being: being potentially and being actually. Since the application of a dichotomy between actuality and potentiality, they argue, is not confined to any one category, it is likewise impermissible to seek to restrict δύναμις to only one of Aristotle's ten categories or to a certain kind of being, i.e., to internal causal power. In their view, thus, capacity or the inactive causal power cannot be identified with δύναμις itself, but rather it is causal power in a potential mode, just as incomplete substances are substances in potential mode. Inactive causal power and incomplete substances are two different kinds of potential beings.

If the unitarians are right, it becomes impossible to assimilate mathematical objects to potentialities or beings in potentiality. But what if the dualists are right? The dualist interpretation could provide room for geometrical objects in Aristotle's ontological space. If the matter of mathematical objects can be seen as potentiality in the sense of another mode of being, mathematical objects could be said to exist in that sense. The question, though, is whether the antecedent is true.

When we compare pure extension with any incomplete substance widely taken as

[342] For the unitary approach, see Frede (1994); Gill (1989).

[343] Ross (1924) Vol. I, pp. cxxiv-cxxv.

an example of another mode of being, evident differences, though, between the mathematical objects' matter and such incomplete substances force themselves on our attention. One is that pure extension lacks the internal capacity or causal power to change itself or another thing into its actuality (supposing that geometrical objects have any actuality); for instance, while a boy by himself will become a man "unless something prevents and hinders it," pure extension will remain shapeless until our intellect imposes thereon a certain geometrical form.

Since pure extension lacks the internal causal power to actualize itself into a geometrical object, its causal relationship with geometrical objects is not obvious. In contrast, a strong causal relationship obtains between an incomplete substance and its actuality. For instance, the incomplete form of a man immanent in a boy realizes itself if nothing hinders its realization, and in the process of its realization, arranges or organizes all the matter, such as bones and flesh, in such a way that its form is fully realized. In this sense, the incomplete form of a man in a boy can be seen as the efficient cause which develops the boy into a man.[344] The incomplete form in a boy also provides a man with its formal cause in that, once the incomplete form is fully realized, it becomes the form of a man. And the complete form of a man is the final cause of a

[344] In *Met*., VIII, 3, 1044a35, Aristotle identifies the efficient cause of a man with sperm. This seed can be understood as the efficient cause of a man in that the sperm carries the form of the man's father. Conception occurs when the sperm imparts form to the menstrual fluid, which is regarded as the matter of man by Aristotle (*Ibid*., 1044a35). In this sense, that by which a man comes into being is the form which the sperm transfers to the menstrual fluid. While the sperm is the carrier of form in the process of conception, thereafter, it develops into a living fetus and thereafter a boy who carries the spermatic form through a long process of growth and development. Thus, it seems to be reasonable to call the boy the efficient cause of a man as well.

boy;[345] the boy exists for the sake of being a man, and a boy becomes a man when his incomplete form is fully actualized.[346] Finally, a boy also provides the material cause of a man in the sense that the form of a man is realized on the body of a boy.[347] In these terms, it might be said that a boy contributes to every cause of the existence of a man. While a boy is the cause of a man, it also holds good that a man is the cause of a boy; the potential form of a man in a boy is contributed by a man, which plays the role of every cause other than the material cause. As Aristotle says in several places, "a man begets a man."[348]

The second difference between pure extension and incomplete substances is even more important. One important Aristotelian thesis of the relationship between actuality and potentiality is that actuality is *prior* to potentiality. Aristotle argues that actuality is prior to potentiality in all of definition, time, and being.[349] It is prior in definition: Any account of 'being capable to be F' always involves the account of 'being F' since 'being capable to be F' cannot be defined without reference to 'being F.' Actuality is prior in time; e. g., a man produces a boy, and an oak tree produces an acorn. Finally, actuality is

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[345] For Aristotle, in the case of a natural object, efficient, formal, and final causes usually coincide. For the coincidence of the three causes, see *Phys*., II, 7, 198a25-30; 8, 199a31-33; *DA*., II, 4, 415b9-15; *GA*., 1, 715a7-11.

[346] Witt argues that the teleological relationship between a potential being and its actuality is an indication that "Aristotle constructs a developmental, hierarchical, and intrinsically normative concept of being." See Witt (2003) p. 3.

[347] Aristotle regards the body as matter of man (*DA*., II, 1, 412a15-22; *Met*., VII, 11, 1037a6, etc.).

[348] *Phys*., III, 2, 202a11-12; *Met*., XII,3, 1070a28; *Pol*., I, 6, 1255b1-2, etc.

[349] For Aristotle's remarks on the priority of actuality to potentiality in time, see *Met*., IX, 8, 1049b23-26; 1049b19-23. Potentiality is, however, prior to actuality in time in some sense; e.g., a boy becomes a man. Priority in time is the only kind of priority which Aristotle assigns to potentiality over actuality. For this priority, see *DA*., II, 3, V, 11, 414b28-32; *Met*., 1019a6-10.

prior in being. For Aristotle, A is prior to B in being if and only if B cannot exist without there being A, not the other way around. In this sense, potentiality is ontologically dependent on actuality, e. g., a boy can exist only if its father existed before. This thesis of the threefold priority of actuality to potentiality is called 'Aristotelian actualism.'[350] However, Aristotelian actualism would not seem to pertain in any obvious way to geometrical objects. While Aristotle's actualism suggests that actuality should be prior to its potentiality in being, the matter of geometrical objects, pure extension, exists as a property of sensible objects regardless of whether geometrical figures exist or not. But it is obvious that geometrical figures cannot exist without pure extension (if they can be said to exist in any sense), in particular if we accept the view that geometrical figures are constructed by the intellect; the intellect constructs a new geometrical figure by imposing its form on pure extension. Thus, if pure extension is the potentiality of a geometrical figure, the potentiality of a geometrical figure ontologically should precede its actuality. But if we want to posit Aristotelian actualism as a general principle applying to any kind of potentiality and actuality, supposing that Aristotle is consistent in his actualism, the ontological priority of pure extension to geometrical figures could be a ground for doubting that the matter of geometrical objects, i.e., pure extension, is legitimately a kind of Aristotelian potentiality.

However, in Aristotle, not every kind of potentiality has the capacity to changes itself or other things into its actuality, nor is posterior to its actuality in being. Aristotle extends the scheme of potentiality-actuality to the relationship between an artifact and its

[350] Witt (2003) p. 15. This actualism is distinguished from Megarian actualism mentioned above.

matter. He explains that:

> It seems that what we are talking about is not 'this' but 'of that' (e.g. a casket is not wood but of wood, and wood is not earth but of earth, and again perhaps earth, if it is in the same way, is not something else than 'of that'), 'that' which is always absolutely in potentiality is what is posterior, e.g., a casket is not earthen nor earth, but wooden; for wood is potentially a casket and is the matter of a casket, wood in general for casket in general, and this particular wood for this particular casket.
>
> ἔοικε δὲ ὃ λέγομεν εἶναι οὐ τόδε ἀλλ᾽ ἐκείνινονοἷον τὸ κιβώτιον οὐ ξύλον ἀλλὰ ξύλινον, οὐδὲ τὸ ξύλον γῆ ἀλλὰ γήϊνον, πάλιν ἡ γῆ εἰ οὕτως μὴ ἄλλο ἀλλὰ ἐκείνινον ἀεὶ ἐκεῖνο δυνάμει ἁπλῶς τὸ ὕστερόν ἐστιν. οἷον τὸ κιβώτιον οὐ γήϊνον οὐδὲ γῆ ἀλλὰ ξύλινον: τοῦτο γὰρ δυνάμει κιβώτιον καὶ ὕλη κιβωτίου αὕτη, ἁπλῶς μὲν τοῦ ἁπλῶς τουδὶ δὲ τοδὶ τὸ ξύλον.[351]

Although Aristotle distinguishes artificial production from natural generation,[352] the production of an artifact is also regarded as a sort of change in that it is a transition from something to another thing, e.g., from unformed bronze to a bronze statue or from the materials of a house to a house. Thus, in Aristotle's view, the matter of an artifact stands as the potentiality of the actual artifact.[353]

However, the matter of artifacts differs from the two other kinds of potentiality we considered above: potentiality as an inactive causal power and potentiality as a way of existence. One distinguishable feature of artifacts is that their potentiality is ontologically prior to their actuality. Consider any artifact. It is obvious that, for instance, before the

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[351] *Met.*, IX, 7.1049a19-24.
[352] *Ibid.*, VII, 7.
[353] See Makin (2006), p. 158

first airplane existed, its material previously existed; the existence of the airplane was dependent on its material, but not *vice versa*. It might be argued that a form of an artifact 'exists' prior to its instantiation in the mind of the artisan. But there would be two responses to this objection. First, while most artifacts were not even conceived by any artisan before they were invented, it is reasonable to say that artifacts' matter existed before the artifacts were specifically conceived, e.g., the material of airplanes existed before the invention of airplanes. Second, a form of an artifact in the mind of an artisan can be said to exist only potentially in the sense that it is not yet physically actualized. If we take an artifact's form in the mind of the artisan as a kind of potential being, we run into the same problem that the potentiality of an artifact is prior to its actuality.[354] Another difference from other kinds of potentiality is that the artifacts' potentiality is not endowed with the capacity in itself whose exercise yields its actuality. While a boy has the immanent causal power to change himself into a man should nothing external prevent, the material of a house never becomes a house by itself—it requires an external agent to exercise such a building capacity. Since the material of artifacts lacks this kind of inner causal power, no causal relationship connects artifacts and their matter, either. For instance, the efficient cause of a house does not lie in the material of the house, but in the artisan who builds the house; the final cause of a house is this artisan's end or goal, and

[354] Hintikka, however, argues that the form in the mind of the artisan is *mentally actualized* when the artisan thinks of it and the mental actualization does not ontologically differ from physical actualization for Aristotle. I examine Hintikka's view on mental actualization and its ontological implications in Chapter Four, §4, aiming to show that mental actualization is ontologically different from physical actualization.

the formal cause of a house its form in the artisan's mind. [355] Likewise, houses do not make a contribution to there being the material of a house; the material of a house pre-exists the house. Except for the fact that the matter of a house provides the material cause of the house, there is no causal connectivity between artifacts and their material.

In this regard, the matter of a mathematical object is more analogous to the matter of artifacts than to the potentiality of a natural thing. We distinguished the matter of geometrical figures from other kinds of potentialities on the grounds that (i) it is prior to geometrical objects in being, (ii) absent the inherent causal power to actualize itself, this matter does not have a causal relationship with geometrical objects except insofar as it provides their material cause. Since the forms of geometrical objects are externally imposed on pure extension by the intellect in constructing figures, the only causal contribution of pure extension is to provide geometrical figures with their matter.

Notice that the matter of arifacts shares these characteristics with that of mathematical objects; yet Aristotle is willing to regard this first matter as a kind of potentiality. Thus, if the matter of an artifact is regarded as a kind of potentiality by Aristotle, the differences between the matter of mathematical objects and other modes of potentiality should not, then, be construed as a sufficient basis on which to disqualify the matter of geometrical objects as a sort of potentiality. If the matter of artifacts is their potentiality, we could plausibly say that the matter of geometrical figures is their potentiality as well.

[355] While there is a teleological relationship between the potentiality and actuality of a natural substance, i.e., a boy exists for the sake of being a man, such a relationship does not obviously hold between an artifact and its matter—or at least, it is not intrinsic.

But this does not get us anywhere. Even if we grant that pure extension could be the potentiality of geometrical objects, just as the matter of artifacts constitutes their potential being, the question of whether mathematical objects can be seen as possessing a sort of existence in Aristotle's ontology remains undecided. This is because potentiality has several distinct senses and we have not yet determined the sense in which the matter of an artifact is its potentiality.

While Aristotle says that being potentially is another meaning of being,[356] he declines to define the term 'potentiality', only suggesting that it may be grasped by analogy.[357] For instance, it could be case that Aristotle calls the matter of artifacts 'potentiality,' not thereby meaning another mode of existence of artifacts, but only using the term to indicate the starting-point of a certain transition, namely the production of an artifact. There are good grounds for accepting this latter position. Let us, first, consider following sentences:

(1) A boy is actualized into a man.

(2) A capacity of seeing (sight) is actualized into seeing.

While these assertions are intuitively attractive, there seems something deficient in:

(3) Timber is actualized into a ship.

[356] *Phys.*, III, 6, 206a1 4-15; *Met.*, IX, 1, 1045ba32-35; VI, 2, 1026b2.

[357] *Met.*, IX, 6, 1048a35-1048b9.

Rather, we seem to have to say:

> (4) The form of a ship in the mind of the shipbuilder is actualized into the ship he builds.

In other words, the relationship between the matter of an artifact as potentiality and its actuality is different form that between an incomplete substance and its actuality, or that between capacity and its actuality, namely, unlike that holding between ‘a boy’ and ‘a man’ or between ‘sight’ and ‘seeing,’ ‘timber’ and ‘a ship’ is not related by the relationship of ‘being actualized into.’ In fact, what is actualized as an artifact is not the matter or something in the matter but rather its form in the artisan’s mind by the medium of its matter. Thus, if an artifact has any potentiality whose relation to its actuality can be likened to the relation of an incomplete substance to its complete development, or to the relationship of a capacity and its exercise, it should be the form of the artifact in the mind of the artisan rather than its matter. So, we can say, at least, that the matter of an artifact is not called potentiality in the same sense that applies to an incomplete substance or a capacity.

Secondly, it is not clear whether even actual artifacts can be regarded as real entities in Aristotle’s metaphysics. If they are something existent, they must belong to one of Aristotle’s categories; this category should be substance since a particular artifact is what is neither said of any subject nor is in any subject; rather, it takes the place of a subject in a sentence.[358] Nevertheless, in several places in the *Metaphysics*, we see Aristotle question the substantial status of artifacts.[359] And there are good reasons to think that, for Aristotle, artifacts are not substances. First of all, while every substance has its own form, it is questionable whether artifacts do so in any proper sense. In *Physics*, II, 1, Aristotle draws a distinction between artifacts and living things by appealing to differences between them: living things have the principle of change in themselves, but the principle of change for artifacts exists in something else than themselves (ἐν ἄλλῳ).[360] Importantly, Aristotle identifies this internal principle of change with nature and nature with the form specified in the definition of the thing.[361] As mentioned earlier,

[358] For Aristotle's criterion of substance in *Categories*, see *Cat*., 5. In *Met.*, VII, Aristotle suggests other criteria of substances as well: separateness, thisness and priority (*Ibid.*, 3, 1029a27-33; 13, 1038b23-9). Further, he identifies forms with substances in that they satisfy these criteria. Even if we apply these criteria to artifacts, artifacts remain disqualified as substances insofar as they lack proper forms.

[359] *Met*., III, 4, 999b17-20; VIII, 3, 1043b18-23; XI, 2, 1060b23-28; XII, 3, 1070a13-18, etc.

[360] *Phys*., II, 1, 192b13-32.

[361] *Ibid*., 193a28-31; 193b6. It is controversial whether the internal principle of change includes the principle of generation or whether they are two distinct principles. Katayama argues that the principle of generation is not an instance of the internal principle of change on the grounds that nothing generates itself (Katayama (1999) p. 110). But Aristotle on more than one occasion identifies the principle of generation with nature (φύσις) (*Met*., XII, 3, 1070a6-7). That is, 'the internal principle' and 'the principle of generation' have the same reference, namely, nature.

Making a distinction between the internal principle of change and that of generation, Katayama also maintains that only the principle of generation can be considered to be a criterion of substantiality (Katayama (1999) p. 110); by having an internal principle of generation, it is possible for living things to perpetuate their own species (*Ibid*., p. 104-105). He points out that mules and spontaneous organisms, according to Aristotle, are not substances in spite of their having the internal principle of change (*Ibid*., p. 104). Living things are considered substances not because they have the principle of change in themselves but because they can perpetuate their own species (*Ibid*., p. 104). Likewise, artifacts are not substantial insofar as they lack the principle of generation in themselves. In the case of artifacts, the principle of generation is art for Aristotle, in the case of living things, "the form (τὸ εἶδος), the mover (τὸ κινῆσαν), and that for the sake of which (τὸ οὗ ἕνεκα) often coincide."[362] The form of a living thing is its moving cause: For instance, it is the soul of a boy which in the process of a boy's growing arranges or organizes all his matter in such a way that the soul of the body can fully function. Once the soul is fully actualized, it changes and moves the body so as to maintain its full functioning. In these terms, the soul is the principle of change in an animal (which Aristotle also understands as its form). Since, for Aristotle, living things are substances most of all (μάλιστα οὐσία),[363] whose internal principles are identified with their forms, the forms of artifacts, e.g., the shape of a bronze sphere, should be distinguished from the forms of substances. Namely, artifacts do not have proper forms or have forms only by analogy.[364] Since the forms of artifacts differ from

(τέχνη), which exists externally in something else, i.e. in the mind of artisans (*Met.,* XII, 3, 1070a7). Thus, the existence of an artifact is dependent upon the existence of an artisan; in contrast, living things are ontologically dependent only on their own species.

362 *Phys*., II, 7, 198a25-30. For the coincidence of efficient, formal, and final causes of living things, see Chapter Four, §3, 3.3, n.345.

363 There are several passages that explicitly or implicitly support the view that all living things are substances in the highest sense: *Cat*., 5, 2a35-2b6; *Met*., VII, 7, 1032a18-19; 8, 1034a3-4; 17, 1041b28-31; VIII, 3, 1043b22-23; XII, 3, 1070a17-19, etc.

364 Another reason to doubt the existence of forms of artifacts is that the forms of artifacts are created by the intellect, whereas the forms of substances are not generated but rather eternal. Thus, strictly speaking, we might say there is no form of an artifact.

Aristotle does not explicitly deny the existence of forms of artifacts. But he agrees with Plato that artifacts do not have their Forms, if there are Platonic Forms at all. He says, for instance, "Now in some cases the 'this' does not exist apart from the composite substance, for instance, the form of house does not so exist if not the art of [building] exists (nor is there generation and destruction of these forms, but it is in another way that "house" and "health" and everything made by art do or do not exist without its matter); but in the case of natural objects, they really do so. And so Plato did not say so wrongly that there are as many Forms as there are those of substances (or they do not have forms at all in a strict sense), it can be inferred that they are not substances.[365] And if they are neither substances nor properties of a substance, it is difficult to see how we can find a place for them in Aristotle's ontology. 'House,' for example, does not refer to a substance, but to a heap of material unified in such a way that it functions as a shelter for human beings. If it is accepted that actual

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kinds of natural things, if there are Forms at all (ἐπὶ μὲν οὖν τινῶν τὸ τόδε τι οὐκ ἔστι παρὰ τὴν συνθετὴν οὐσίαν, οἷον οἰκίας τὸ εἶδος, εἰ μὴ ἡ τέχνη (οὐδ' ἔστι γένεσις καὶ φθορὰ τούτων, ἀλλ' ἄλλον τρόπον εἰσὶ καὶ οὐκ εἰσὶν οἰκία τε ἡ ἄνευ ὕλης καὶ ὑγίεια καὶ πᾶν τὸ κατὰ τέχνην), ἀλλ' εἴπερ, ἐπὶ τῶν φύσει: διὸ δὴ οὐ κακῶς Πλάτων ἔφη ὅτι εἴδη ἔστιν ὁπόσα φύσει, εἴπερ ἔστιν εἴδη) (*Met*,. XII, 3,1070a13-19)." We do not know the exact rationale for Plato's denial of the existence of Forms of artifacts. But, as the passage suggests, forms of natural things are not subject to generation and destruction, while the Forms or forms of artifacts, should there be any such, are conceived by artisans at a certain time. Since both Aristotelian forms and Platonic Forms are eternal, this difference could underlie the denial of the existence of Forms and forms of artifacts. Aristotle at certain points talks about forms of artifacts but in a different sense from that in which natural things exist.

365 Scholars have suggested three different reasons for the denial of the substancehood of artifacts on the basis of differences between living things and artifacts: (i) Artifacts do not have the internal principle of change of living things (Sellars (1967) pp. 78, and 119-124; Lewis (1994) pp. 263-265) (ii) the form of an artifact does not possess function which the form or soul of a body has (Gill (1989) pp. 161, 213, 221, and 242; Irwin (1988) pp. 571-572), and (iii) the relation between the matter of artifacts and their forms lacks the intrinsic unity of the soul and the body (Gerson (1984); Halper (1989); Kosman (1987); Ferejohn (1994); Block (1978)). I take (i) to be one reason to deny the existence of forms of artifacts. Katayama, however, argues that (i), (ii), and (iii) are not strong enough to deprive artifacts of the status of substance. He points out that those views are based on the assumption that living things are substances most of all. But, he argues, this is dubious because there is no evidence that Aristotle considers mules and spontaneously generated organisms as anyhow substances (see Katayama (1999) pp. 18-23). For the argument against the substantiality of mules, see also Rorty (1973) pp, 393-420). I will not further discuss each of (i), (ii) and (iii) in detail (for a brief survey of these three views, see Katayama (1999) pp. 122-124); even if artifacts are substances and their matter a kind of existence, as we will see, there are other good Aristotelian reasons to deny that the matter of a geometrical figure is a kind of existence—i.e., there is no actuality of geometrical figures.

artifacts are not real units of existence, *a fortiori* their potentialities cannot be seen as some kind of existent.

If the matter of artifacts is a kind of potentiality but not in the sense of existence, it would be a wasted effort to try to show that the matter of a geometrical figure is its potentiality on the basis of similarities between the matter of geometrical objects and that of artifacts. But what if artifacts can be seen as substance at least with some qualification? If artifacts are substances, there could be a possibility that the matter of artifacts is their potentiality *qua* another mode of the existence of artifacts. Even if we accept the matter of artifacts as a kind of existence, however, this would not by itself guarantee that the matter of a geometrical figure is also another mode of existence. It should be noted that Aristotle regards matter as potentiality because every change is a transition from some potentiality to some actuality, and the matter of an artifact is changed into an actual artifact. This is what allows Aristotle to calls the matter of an artifact 'potentiality', in spite of his own actualism. Unlike artifacts, however, many geometrical objects do not have actual physical instantiations; if there is no actuality of some geometrical objects, there can be no change or transition from their matter to their actuality. Then, whatever their matter is, there is no basis for saying that their matter is the potentiality of actual geometrical objects. More generally, it is questionable how something can be only in potentiality without the presence of a corresponding actuality. If we consider Aristotle's emphasis on the priority of actuality in existence, it is not likely that he could accept that something can exist not in actuality but only in potentiality. Aristotle's ontology would not seem to countenance the existence of potential beings independently from actual beings. Thus, insofar as Aristotle maintains his actualism, there cannot be something which exists only in potentiality. This implies that insofar as some geometrical figures do not exist in actuality, they cannot exist in potentiality, either.

3.4. Infinity-Potentiality without Actuality

However, Aristotle does seem to hold out the prospect of something which is in potentiality and is never actualized. He says, "Some things exist only actually, some potentially, some potentially and actually—some as substances, some quantities, others in the other categories (῎Εστι δὲ τὸ μὲν ἐνεργείᾳ μόνον τὸ δὲ δυνάμει τὸ δὲ δυνάμει καὶ ἐνεργείᾳ, τὸ μὲν ὂν τὸ δὲ ποσὸν τὸ δὲ τῶν λοιπῶν)."[366] We should again begin by clarifying the sense he uses the term 'potentially' here. We may then proceed to determine whether the term, 'potentiality', in this sense, means a certain kind of existence.

Aristotle's account of *infinity* sheds light on this matter; the infinite is an example of a potentiality that is not actualized. For Aristotle, the infinite is not in actuality. He says, "it is clear that the infinite cannot exist actually (ὅτι οὐκ ἔστιν ἐνεργείᾳ εἶναι τὸ ἄπειρον, δῆλον)."[367] First, there cannot be actual infinite extension.[368] In Aristotelian cosmology, the universe is finite in its size, and outside the universe nothing

[366] *Met.*, XI, 8, 1065b5-7.
[367] *Ibid.*, 1066b11-12
[368] *Phys.*, III, 5, 204b5-6; *DC.*, I, 5, 271b17-273a24.

extended can exist; Aristotle denies the existence of any empty space which exists by itself independently of all bodies.[369] Secondly, Aristotle further denies the existence of an infinite number. He does so on two grounds: (i) he presupposes that a number is something countable. But since infinite number is not countable, *ex hypothesi* there is no infinite number.[370] (ii) Aristotle also argues that the infinity cannot exist separately in itself, since it does not exist as a substance but only as a property of a certain kind of quantity, such as extension or number.[371] But since there is no infinite number or extension in the actual world, the infinite cannot exist in actuality, either.[372]

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[369] *DC*., I, 1, 268b8-10

[370] *Phys*., III, 5, 204b6-8.

[371] *Met*., XI, 8, 1066b9-14. See also Hussey (1983), pp-78-80.

[372] Aristotle's account of infinity raises a couple of problems with regard to time and mathematics. The first is that, according to Aristotle, there is no beginning of time (for Aristotle's rejection of a beginning or an end of time, see *Phys*., IV, 13, 222b6-7; VIII, 1, 251a8-252a5); since there is no beginning of time, there must be an *infinite* span of time. Whatever time unit we use to measure time's span, i.e., whether we count it by year or day, we must attribute an infinite number to its temporal extent. Thus, it might be argued that an infinite number exists as a quantitative property of time. Aristotle nowhere deals with this problem, but would have to avail himself of one or both of two possible responses: (i) The future will exist and the past existed, but only the present *exists*. Thus, an infinite span of time does not exist actually. (ii) Time is a kind of number and numbers as mathematical objects are fictional entities, in the sense that their existence depends on our mental operations such as counting. For Aristotle, "it is impossible for there to be time unless there is soul, if nothing but soul, or in soul reason, is qualified to count (*Phys*., IV, 14, 223a25-26)." Since time is fictional, the infinity which is its property should also be fictional. (For the fictionality of numbers, see Appendix). The second problem is that the concept of infinity is current in the geometry of Aristotle's day. For instance, *Elements* I, Def. 23, Post 5, Prop. 29 speaks of lines being extended infinitely. But Aristotle believes that such infinite lines might be replaced with finite line sections without harming the practice of geometry. He says, "This account does not deprive the mathematicians of their study, by disproving that there is actually the infinite such that it is untraversable in the direction of increase; for, as a matter of fact, they do not need the infinite (for they do not use it), but they need only that there should a finite line of any length they wish. It is possible to have divided another magnitude of any size whatever in the same ratio as the largest quantity. Hence, for the purpose of proof, it will make no

Nevertheless, Aristotle argues that the infinity *is* in some sense, namely, it *is* in potentiality.

> It is clear that in a sense [the infinite] is and in a sense it is not. 'To be', then, means either 'to be in potentiality' or 'to be in actuality'; and the infinite is, on the one hand, in addition and, on the other hand, in division. It has been stated that that magnitude is not actually infinite. But it is infinite in division; for it is not hard to refute indivisible lines. Then, it remains for the infinite to be in potentiality. And we must not take 'being in potentiality' in the same way as that in which, if it is possible for this to be a statue, it will be a statue in actuality; something will be infinite in actuality not in this sense. 'To be' has many senses, just as the day is, and the contest is, i.e., one thing after another always occurs; so too with the infinite. (In these cases too there is 'in potentiality' and 'in actuality'; for there are the Olympic games, both in the sense that the contest is able to occur and in the sense that it is occurring.)
>
> δῆλον ὅτι πὼς μὲν ἔστιν πὼς δ᾽ οὔ. λέγεται δὴ τὸ εἶναι τὸ μὲν δυνάμει τὸ δὲ ἐντελεχείᾳ, καὶ τὸ ἄπειρον ἔστι μὲν προσθέσει ἔστι δὲ καὶ διαιρέσει. τὸ δὲ μέγεθος ὅτι μὲν κατ᾽ ἐνέργειαν οὐκ ἔστιν ἄπειρον, εἴρηται, διαιρέσει δ᾽ ἐστίν: οὐ γὰρ χαλεπὸν ἀνελεῖν τὰς ἀτόμους γραμμάς: λείπεται οὖν δυνάμει εἶναι τὸ ἄπειρον. οὐ δεῖ δὲ τὸ δυνάμει ὂν λαμβάνειν, ὥσπερ εἰ δυνατὸν τοῦτ᾽ ἀνδριάντα εἶναι, ὡς καὶ ἔσται τοῦτ᾽ ἀνδριάς, οὕτω καὶ ἄπειρον ὃ ἔσται ἐνεργείᾳ: ἀλλ᾽ ἐπεὶ πολλαχῶς τὸ εἶναι, ὥσπερ ἡ ἡμέρα ἔστι καὶ ὁ ἀγὼν τῷ ἀεὶ ἄλλο καὶ ἄλλο γίγνεσθαι, οὕτω καὶ τὸ ἄπειρον (καὶ γὰρ ἐπὶ τούτων ἔστι καὶ δυνάμει καὶ ἐνεργείᾳ: ᾿Ολύμπια γὰρ

difference to them that [the infinite] is among existent magnitudes (οὐκ ἀφαιρεῖται δ᾽ ὁ λόγος οὐδὲ τοὺς μαθηματικοὺς τὴν θεωρίαν, ἀναιρῶν οὕτως εἶναι ἄπειρον ὥστε ἐνεργείᾳ εἶναι ἐπὶ τὴν αὔξησιν ἀδιεξίτητον: οὐδὲ γὰρ νῦν δέονται τοῦ ἀπείρου (οὐ γὰρ χρῶνται), ἀλλὰ μόνον εἶναι ὅσην ἂν βούλωνται πεπερασμένην: τῷ δὲ μεγίστῳ μεγέθει τὸν αὐτὸν ἔστι τετμῆσθαι λόγον ὁπηλικονοῦν μέγεθος ἕτερον. ὥστε πρὸς μὲν τὸ δεῖξαι ἐκείνοις οὐδὲν διοίσει τὸ [δ᾽] εἶναι ἐν τοῖς οὖσιν μεγέθεσιν) (*Phys.,* III, 7, 207b28-35) ." For Hussey's discussion of Aristotle's finitism, see Hussey (1983), pp. xx-xxvi; 93-96; 178-179.

ἔστι καὶ τῷ δύνασθαι τὸν ἀγῶνα γίγνεσθαι καὶ τῷ γίγνεσθαι)[373]

Aristotle maintains that it is not possible to produce an infinite magnitude by addition, on the ground that that there can be no actual infinite magnitude. This inference is possible because he also grants elsewhere that the size of universe is fixed;[374] were the universe continually to expand, there could at least be a potential infinity in some sense. Although Aristotle denies the possibility of a potential infinity in respect of continual addition, he does argue that a potential infinity can be generated by repeatedly dividing a certain magnitude. A series of magnitudes can be produced by dividing a magnitude e.g. a finite line *ad infinitum* (given that Aristotle rejects the existence of the indivisible line).[375] The series is potentially infinite in that the process which increases the number of its members can be continued endlessly. Nevertheless, the series is only potentially infinite because its size is always only finite at the moment of any dividing operation, meaning that the actualization of the potentiality of the infinite remains a mere possibility.[376]

This potentiality is different from the potentialities which we examined previously. We considered three different meanings of potentiality: (i) potentiality as

[373] *Phys*., III, 6, 206a13-25.

[374] *Ibid.,* 7, 207b15-21. See also Hussey (1983) pp. 84-85.

[375] *Phys*., I, 5, 188a12.; VI, 1, 231a21-231b6; 231b16-18; 2, 232a23-25. Aristotle's view is in accordance with the practice of geometry of his day. In Euclidean geometry, every line can be divided into two. And if geometry is applicable to physical objects, any physical extension should be divisible *ad infinitum.* For the Aristotle's criticism of the concept of indivisible extension, see *Ibid*., VI, 1, 231b15-232a22.

[376] Hussey (1983) p. xxiii.

inactive causal power, (ii) potentiality as another mode of existence, and (iii) potentiality as the matter of substances or substance-like things, i.e., artifacts. Whatever their differences, these modes of potentiality are alike in that each has its corresponding actuality, bound in strong or weak causal relationships to its potential; further, it is true of all that these potentialities "are some kind of origins, and are so called in reference to one primary kind, which is a starting point of change in something else or in the thing itself *qua* something else (ἀρχαί τινές εἰσι, καὶ πρὸς πρώτην μίαν λέγονται, ἥ ἐστιν ἀρχὴ μεταβολῆς ἐν ἄλλῳ ἢ ᾗ ἄλλο)."[377] However, the infinite as potentiality neither has its actuality nor exists in any causal relationship with its actuality. Moreover, since there can be no actuality of the infinite, the sense in which it is in potentiality cannot be the primary sense of the term. A potentiality is regarded as a starting point of some change when the change is understood as the actualization of that potentiality. But since there is no actuality of the potentiality of the infinite, the potentiality of the infinite is not in this way a starting point of change. Aristotle phrases this point in the following way.

> …what is in the primary sense potential is potential because it is possible for it to be actualized, e.g., I mean by 'capable of building' that which can build, and by 'capable of seeing' that which can see, and by 'visible' that which can be seen.
>
> γὰρ ἐνδέχεσθαι ἐνεργῆσαι δυνατόν ἐστι τὸ πρώτως δυνατόν, οἷον λέγω οἰκοδομικὸν τὸ δυνάμενον οἰκοδομεῖν, καὶ ὁρατικὸν τὸ ὁρᾶν, καὶ ὁρατὸν τὸ δυνατὸν ὁρᾶσθαι.[378]

[377] *Met.*, IX, 1, 1046a9-11.
[378] *Met.,* IX, 8, 1049b13-16.

Thus, it can be concluded that, while Aristotle has a concept of potentiality which does not entail the existence of its corresponding actuality, this type of potentiality is quite different from other kinds of potentiality. Then, even if other kinds of potentialities can be seen as involving a mode of existence, the non-actualizable potentiality of the infinite does not need to be some kind of existent. Rather, considering Aristotle's emphasis on the priority of actuality to potentiality in existence, it is very unlikely that Aristotle will view non-actualizable potentiality as an existent entity.

Aristotle's texts endorse this inference. We must remember that 'being potentially' has many senses for Aristotle, not all of them necessarily entailing 'existence'. While 'being potentially' on the one hand may denote a mode of existence, on the other, it may also signify a kind of non-existence:

> The term 'actuality', which we connect with 'reality', has been extended primarily from changes to other things; for actuality in the primary sense is thought to be identical with change. Hence people do not assign change to non-existent things, though they do assign some other predicates. For instance, they say that non-existent things are conceivable and desirable, but not that they are changed; and this is because, while they do not actually exist, they will exist actually [if they were changed]. For of non-existent things some exist potentially; but they do not exist, because they do not exist in actuality.
>
> ἐλήλυθε δ᾽ ἡ ἐνέργεια τοὔνομα, ἡ πρὸς τὴν ἐντελέχειαν συντιθεμένη, καὶ ἐπὶ τὰ ἄλλα ἐκ τῶν κινήσεων μάλιστα: δοκεῖ γὰρ ἡ ἐνέργεια μάλιστα ἡ κίνησις εἶναι, διὸ καὶ τοῖς μὴ οὖσιν οὐκ ἀποδιδόασι τὸ κινεῖσθαι, ἄλλας δέ τινας κατηγορίας, οἷον διανοητὰ καὶ ἐπιθυμητὰ εἶναι τὰ μὴ ὄντα, κινούμενα δὲ οὔ, τοῦτο δὲ ὅτι οὐκ ὄντα ἐνεργείᾳ ἔσονται ἐνεργείᾳ. τῶν γὰρ μὴ ὄντων ἔνια δυνάμει ἐστίν: οὐκ ἔστι δέ,

ὅτι οὐκ ἐντελεχείᾳ ἐστίν.[379]

Let us consider this argument in greater detail. At first glance, it does not appear to be valid. Aristotle's claims may be summarized as follows:

1. Actuality is identical with change.
2. Non-beings do not change.[380]
3. Some potentialities are not actualized.
4. Those non-actualizable potentialities do not exist.

Aristotle begins the argument with identifying actuality with change.[381] But he takes

[379] *Ibid.,* 1047a30-47b2. Aristotle argues the same point also in *Phys*., V, 1, 225a20-27, *Met*., XIV, 2, 1089a27-30. I translate 'ἐντελέχειαν' as 'reality' to distinguish it from 'ἐνέργεια.' But in Aristotle's works both terms equally mean 'actuality,' and are typically used usually without distinction. For an account of Aristotle's usage of these terms, see Chapter Three, §5, n. 206.

[380] Also see *Phys*., V, 1, 225a20-29.

[381] "The actuality of that which potentially is, *qua* such, is change, e.g., the actuality of that which is alterable, *qua* alterable, is alteration; of that which is increasable and its opposite, decreasable (for there is no common term for both), it is increase and decrease; of what can come to be and pass away, it is coming to be and passing away; of what can be carried along, it is locomotion. And so it is, thus, clear that this is change; for when that which is buildable is in actuality, insofar as we call it such, it is being built, and this is the act of building; and similarly with learning and healing and rolling and jumping and maturing and growing old (ἡ τοῦ δυνάμει ὄντος ἐντελέχεια, ᾗ τοιοῦτον, κίνησίς ἐστιν, οἷον τοῦ μὲν ἀλλοιωτοῦ, ᾗ ἀλλοιωτόν, ἀλλοίωσις, τοῦ δὲ αὐξητοῦ καὶ τοῦ ἀντικειμένου φθιτοῦ (οὐδὲν γὰρ ὄνομα κοινὸν ἐπ᾽ ἀμφοῖν) αὔξησις καὶ φθίσις, τοῦ δὲ γενητοῦ καὶ φθαρτοῦ γένεσις καὶ φθορά, τοῦ δὲ φορητοῦ φορά. ὅτι δὲ τοῦτο ἔστιν ἡ κίνησις, ἐντεῦθεν δῆλον. ὅταν γὰρ τὸ οἰκοδομητόν, ᾗ τοιοῦτον αὐτὸ λέγομεν εἶναι, ἐντελεχείᾳ ᾖ,

this as a ground for establishing the general claim that changing is not a property of non-beings. However, to use 1 as the premise for 2, we should also assume:

5. Anything actual is something existent.

For Aristotle, there is no doubt about the existence of actual beings; "actuality means existence of the thing (ἔστι ἐνεργεια τὸ ὑπάρχειν τὸ πρᾶγμα)."[382] A more critical question is whether there are some kinds of beings other than actual beings. Since actuality is identified with change, from 3 we can infer that:

6. Some potentialities do not change.

From 6 and 2, however, 4 does not follow; from the fact that any non-being does not change and something, say *a,* does not change, it does not follow that *a* is a non-being; the prime mover would be a counterexample here, in that the prime mover does not change but exists in actuality; indeed it is pure actuality.

To evaluate the original argument fairly, though, we must again consider Aristotle's motivation for introducing the concept of potentiality. He invented the concept to explain how change is possible in the face of Megarian actualism: in order to explain

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οἰκοδομεῖται, καὶ ἔστιν τοῦτο οἰκοδόμησις: ὁμοίως δὲ καὶ μάθησις καὶ ἰάτρευσις καὶ κύλισις καὶ ἅλσις καὶ ἅδρυνσις καὶ γήρανσις) (*Phys*., III, 1, 201a10-19)."

[382] *Met.,* IX, 6, 1048a39-31.

how A becomes B when A is not B, it is necessary to assume the existence of the potentiality for B. For A to be changed into B, there must *exist* something which actualizes itself into B if nothing prevents or hinders it; that thing is the potentiality of B; B comes to exist when its potentiality is actualized. Thus, if there is no actualization of B from A, there is no reason to posit such a thing as the potentiality of being B. And since the potentiality of B will not be actualized, it will not change—in that change is the actualization of some potentiality. But it is assumed that non-changing is a common feature of non-beings. If there is no reason to posit the potentiality of B, and it shares in a common feature of non-beings, namely, non-changing,[383] it seems to be fair to claim that, *stricto sensu*, it does not exist.

Leaving to one side the validity of Aristotle's argument against the *existence* of non-actualizable potentiality, it should by now be clear that he does not accept such a potentiality as something existent in his ontology. This consideration is germane to a discussion of the ontological status of mathematical objects because they too (if they can be called potential in any sense) represent a potentiality which may never be actualized. Since non-actualizable potentiality in general is a kind of non-being for Aristotle, it can be concluded that the potentiality of geometrical objects does not exist, insofar as they are merely potential and will not be actualized.

4. The Mental Actualization of Mathematical Objects

[383] One might wonder whether non-beings can have any properties at all. This kind of assertion, though, could be construed as a negative claim made of a potentiality, that is, as simply meaning that such an entity does not change (rather than having the positive property of non-changing).

Hintikka, at this point, proposes a solution. Hintikka argues that Aristotle allows not only for physical actualization but also for mental or immaterial actualization; and geometrical objects can be actualized not in physical matter but in the intellect. His thesis that there can be an actuality of each geometrical object in thought is very important to our discussion, since the apparent failure of some geometrical objects actually to exist lies at the origin of every problem in Aristotle's philosophy of mathematics.

We can at once note that Aristotle in fact mentions the actualization of a geometrical construction in one of the passages we have already examined:

> The geometrical constructions are discovered in actuality because they (mathematicians) discover [them] by dividing [the given figures]. If they had been already divided, they would have been obvious; but as it is they are in there potentially. Why is the triangle two right angles? It is because the angles around one point are equal to two right angles. Thus, if the line parallel to the side had been drawn up, the reason would have been clear immediately on seeing [it]. Why is there universally a right angle in the semi-circle? Because if three lines are equal, the two which are the base and the one set upright from the centre, it is clear on seeing it to one who knows that. Thus, it is evident that the things in potentiality are discovered by being brought into actuality; the reason is that thinking is the actuality, so that the potentiality is from actuality, and because of this they know by constructing, for the individual actuality is posterior in generation.
>
> εὑρίσκεται δὲ καὶ τὰ διαγράμματα ἐνεργείᾳ: διαιροῦντες γὰρ εὑρίσκουσιν. εἰ δ᾽ ἦν διῃρημένα, φανερὰ ἂν ἦν: νῦν δ᾽ ἐνυπάρχει δυνάμει. διὰ τί δύο ὀρθαὶ τὸ τρίγωνον; ὅτι αἱ περὶ μίαν στιγμὴν γωνίαι ἴσαι δύο ὀρθαῖς. εἰ οὖν ἀνῆκτο ἡ παρὰ τὴν πλευράν, ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί. ἐν ἡμικυκλίῳ ὀρθὴ καθόλου διὰ τί; ἐὰν ἴσαι τρεῖς, ἥ τε βάσις

> δύο καὶ ἡ ἐκ μέσου ἐπισταθεῖσα ὀρθή, ἰδόντι δῆλον τῷ ἐκεῖνο εἰδότι. ὥστε φανερὸν ὅτι τὰ δυνάμει ὄντα εἰς ἐνέργειαν ἀγόμενα εὑρίσκεται: αἴτιον δὲ ὅτι ἡ νόησις ἐνέργεια: ὥστ᾽ ἐξ ἐνεργείας ἡ δύναμις, καὶ διὰ τοῦτο ποιοῦντες γιγνώσκουσιν (ὕστερον γὰρ γενέσει ἡ ἐνέργεια ἡ κατ᾽ ἀριθμόν).[384]

Clearly, Aristotle says here that a potential geometrical construction is 'actualized.' I argued earlier that if geometrical constructions can be mentally constructed, so can geometrical figures. By the same logic, if a potential geometrical construction can be in some way actualized, a geometrical figure should be able to be actualized in the same way. One problem, though, emerges in that, although the matter of geometrical objects, pure extension, is obtained from physical world by means of abstraction, this does not guarantee the reality of a geometrical object. As we have argued before,[385] insofar as there is no ontological or causal relationship between mathematical forms constructed by the intellect and actual beings in the external world, a geometrical object actualized only in the mind will remain a fictional entity, with the result that geometry will lose its objective reality.

Hintikka, however, argues that there is little difference for Aristotle between actualization in physical matter and actualization in the intellect.[386] He develops this

[384] *Met.*, X, 1, 1051a21-33
[385] See Chapter Four, §2.
[386] Hintikka (1996) p. 208.

argument on the basis of Aristotle's claim of an identity between the thinking intellect and the objects of thinking.[387]

> ...the intellect is in a way potentially the objects of thought, but it is actually nothing before it thinks; potentially in the same way as characters are said to be on a tablet on which nothing is actually written; this is what happens in the case of the intellect. And it is itself an object of thought, just as its objects are. For, in the case of those things which have no matter, that which thinks and that which is thought are the same; for contemplative knowledge and that which is known in that way are the same…In those things which have matter each of the objects of thought is present potentially. Hence, intellect will not be present in them (for intellect is a potentiality of such things without their matter), but what can be thought will be present in that kind of thing.
>
> ...δυνάμει πώς ἐστι τὰ νοητὰ ὁ νοῦς, ἀλλ᾽ ἐντελεχείᾳ οὐδέν, πρὶν ἂν νοῇ: δυνάμει δ᾽ οὕτως ὥσπερ ἐν γραμματείῳ ᾧ μηθὲν ἐνυπάρχει ἐντελεχείᾳ γεγραμμένον: ὅπερ συμβαίνει ἐπὶ τοῦ νοῦ. καὶ αὐτὸς δὲ νοητός ἐστιν ὥσπερ τὰ νοητά. ἐπὶ μὲν γὰρ τῶν ἄνευ ὕλης τὸ αὐτό ἐστι τὸ νοοῦν καὶ τὸ νοούμενον: ἡ γὰρ ἐπιστήμη ἡ θεωρητικὴ καὶ τὸ οὕτως ἐπιστητὸν τὸ αὐτό ἐστιν … ἐν δὲ τοῖς ἔχουσιν ὕλην δυνάμει ἕκαστον ἔστι τῶν νοητῶν. ὥστ᾽ ἐκείνοις μὲν οὐχ ὑπάρξει νοῦς (ἄνευ γὰρ ὕλης δύναμις ὁ νοῦς τῶν τοιούτων), ἐκείνῳ δὲ τὸ νοητὸν ὑπάρξει.[388]

[387] The identity between the subject of thinking and the objects of thinking is asserted also at *Met*., XII, 7, 1072b20. He says, "And intellect thinks itself through participation in the object of thought; for it becomes an object of thought in coming into contact with and thinking its objects, so that the intellect and the object of thought are the same. For that which is capable of receiving the object thought, i.e. the substance, is the intellect. And it is actuality which it possesses [this object] (αὑτὸν δὲ νοεῖ ὁ νοῦς κατὰ μετάληψιν τοῦ νοητοῦ: νοητὸς γὰρ γίγνεται θιγγάνων καὶ νοῶν, ὥστε ταὐτὸν νοῦς καὶ νοητόν. τὸ γὰρ δεκτικὸν τοῦ νοητοῦ καὶ τῆς οὐσίας νοῦς, ἐνεργεῖ δὲ ἔχων)."

[388] *DA*., III, 4, 429b30-430a10.

Hinitkka argues as follows: Since, when the intellect thinks something, say, x, it becomes x, thinking x entails the actualization of x. That is to say, x is actualized in the mind of the thinker, when the thinker thinks x. For this reason, the conceivability of x implies the actualizability of x.[389] What differentiates actualization in the mind from actualization outside is only the fact that they take hold in different matter; the matter in which the form of a house is actualized in the mind of a builder evidently differs from the physical material out of which the house may be built. However, the kind of matter in which the form is actualized is not important for actualization (mental or physical), nor is it necessary that a form actualizable in mind should also be actualizable in physical matter; some forms can be actualized in both media, others just in one.[390] Thus, mental actualization is as good as any other actualization.[391]

Hintikka further claims that his own view provides solutions for the problems posed by Aristotle's actualism. We have considered three exceptional cases to Aristotle's actualism: artificial products, infinity, and geometrical objects. In those cases, either potentiality is prior to actuality in existence or something is only in potentiality without being actualized. But according to Hintikka, a form of an artifact in its producer's mind can be seen as the actuality of the artifact, since being thought in the intellect is identical

[389] Hintikka (1996) pp. 208 and 209-211.

[390] Hintikka, however, does not identify in what matter a form is actualized in the intellect. This is problematic because there is no clear textual evidence that Aristotle assumes such mental or immaterial matter. One candidate for this immaterial matter would be the intellect itself. The intellect is potentially whatever it thinks; and we saw a conceptual connection between potentiality and matter. The matter of x is potentially x because it will be x by virtue of receiving the form of x. See *DA*., III, 4, 430a10.

[391] Hintikka (1966) pp. 209-211.

with being actualized for Aristotle. If we grant Hintikka's claim, even the first instance of an artifact can have its actuality prior to its potentiality. Similarly, every potentiality can have its actuality i.e. in mind. In the case of infinity, for instance, insofar as we can conceive of it, infinity is actualized.[392] No potentiality goes without its actuality. Actualism may be consistently applied to geometrical figures as well. With regard to geometrical figures, a problem concerning Aristotle's actualism was such figures existed only in potentiality and in a restricted sense. But since we can construct a geometrical figure in the intellect, it will thereby exist in actuality as well.

However, there are significant differences between mental actualization and physical actualization. First, the former is mind-dependent and, in that sense, lacks the objectivity of the latter. In the light of Aristotle's metaphysical objectivism, it is not likely that they will have the same ontological status. Secondly, on Hintikka's view, a form of an artifact in the mind of an artisan should be regarded as its actuality. However, Aristotle identifies this form with its potentiality, not its actuality; the form is rather actualized by being embodied in some physical matter.[393] This indicates that he

[392] Hintikka assumes that for Aristotle infinity is conceivable although it is never physically actualized. But his assumption is dubious. There are several passages which suggest that any infinite set of magnitudes or numbers is never fully present even in our thought (see *Phys*., III, 4, 203b22-25; 7, 207b10-15). In that sense, we may say that the infinite is not in fact mentally actualized.

[393] "Since some such principle are present in soulless things, and others in things possessed of soul, and in soul and in the rational part of the soul, it is clear that also in the case of potentialities some will be non-rational and some will be accompanied by reason. This is why all arts and all productive knowledge are potentialities; for they are origins of changes in something else or in the things itself *qua* something else (Ἐπεὶ δ᾽ αἱ μὲν ἐν τοῖς ἀψύχοις ἐνυπάρχουσιν ἀρχαὶ τοιαῦται, αἱ δ᾽ ἐν τοῖς ἐμψύχοις καὶ ἐν ψυχῇ καὶ τῆς ψυχῆς ἐν τῷ λόγον

does not treat the mental actualization equivalently to physical, even if he does call a certain mental process 'actualization.'

Given the differences between the two actualizations, it would be legitimate to ask whether something actualized in the mind can have ontologically the same degree of reality of existence as something physically actualized in the external world. Problems immediately arise should we admit something actualized in the mind as being existent. First, such a classification would blur Aristotle's distinction between potentiality and mere possibility.[394] Conceivability is usually understood as possibility in the broadest sense; anything can be conceived unless it involves a logical contradiction. So, if thinking is actualization, every possibility will be taken as an actualizable potentiality. And if being actualized in the intellect implied being in existence, anything conceivable should be able to exist. But this would be the last claim that Aristotle's metaphysics can accommodate; if anything conceivable can exist, not only mathematical objects but also all other fictional entities, Platonic Forms, all kinds of *possibilia, etc.,* should exist in his

ἔχοντι, δῆλον ὅτι καὶ τῶν δυνάμεων αἱ μὲν ἔσονται ἄλογοι αἱ δὲ μετὰ λόγου: διὸ πᾶσαι αἱ τέχναι καὶ αἱ ποιητικαὶ ἐπιστῆμαι δυνάμεις εἰσίν: ἀρχαὶ γὰρ μεταβλητικαί εἰσιν ἐν ἄλλῳ ᾗ ᾗ ἄλλο) (*Met.*, IX, 1, 1046a36-b4)."

394 In fact, Hintikka claims that there is no clear distinction for Aristotle between logical and natural possibility (Hintikka (1973) p.107). Some commentators agree: Among the dualists, Menn and Charlton identify this new meaning of *dunamis* with possibility. See Menn (1994) and Charlton (1989). Sorabji also argues that Aristotle does not distinguish between logical possibility and physical possibility. See Sorabji (1969) pp. 127-135. Some textual evidence supports this interpretation. The concept of '*dunamis*,' which Aristotle analyzes in *Prior Analytics*, I, 13, corresponds to logical possibility. Nevertheless, this apparently narrow specification cannot be taken to imply that Aristotle fails to distinguish between various different kinds of possibility. Rather, it might be the case that Aristotle uses terminology homonymously. For a more general discussion of the relationship between potentiality and possibility, see Freeland (1986) and Knuuttila (1933) Ch. 1.

ontology, insofar as they can be conceived. This contradicts Aristotle's anti-Platonism and his metaphysical objectivism in general.

Nevertheless, a long tradition of interpretation seems to support the view that there is no ontological difference between mental actualization and physical actualization, including physical. These commentators set out two different ways in which things can exist in Aristotle's metaphysics: material and immaterial existence; further, the latter is obtained through mental actualization.

This line of interpretation has a heritage extending back to the Ancient Greek commentators, who were the first to concern themselves with the meaning of Aristotle's claim of the identity between thinking and the objects of thinking. Aristotle maintains in several places that the intellect, the subject of thinking, is potentially the objects of thinking; and when the intellect is in its actuality, i.e. when it thinks an object, it becomes identical with the object.[395] But how can the intellect become its objects? Obviously, this cannot mean that when the intellect thinks of a rock, it becomes physically the rock as such. Then, what are the objects of thinking?

> In the soul, that which can perceive and that which can know are potentially these things, the one the object of knowledge, the other the object of perception. And they must be either the things themselves or their forms. Not the things themselves; for it is not the stone which is the soul, but its form.
>
> τῆς δὲ ψυχῆς τὸ αἰσθητικὸν καὶ τὸ ἐπιστημονικὸν δυνάμει ταὐτά ἐστι, τὸ μὲν <τὸ> ἐπιστητὸν τὸ δὲ <τὸ> αἰσθητόν.

[395] *DA.,* III, 6, 429b30-7, 430a10; 7, 431b7-8, 432a3; *Met.,* XII, 7, 1072b20-23.

> ἀνάγκη δ᾽ ἢ αὐτὰ ἢ τὰ εἴδη εἶναι. αὐτὰ μὲν δὴ οὔ: οὐ γὰρ ὁ λίθος ἐν τῇ ψυχῇ, ἀλλὰ τὸ εἶδος.[396]

This is one of several places where Aristotle identifies the objects of perception and thought with *forms* of things.[397] Now since the forms of things are the objects of thinking, to think of A means for the intellect to become identical with the *form* of A. How does the intellect become such a form? The explanation Aristotle gives in *De Anima* is that the intellect receives the form from the thing to which the form belongs. Likening the process of perception to wax's reception of the imprint of a ring, he says:

> And in general, with regard to all perception, it is necessary to understand that a sense is that which is receptive of sensible forms without the matter, as wax receives the imprint of the ring without the iron or gold, and it takes the imprint of bronze or gold, but not *qua* gold or bronze. Similarly in each case the sense is also affected by that which has color or flavor or sound, but by it not insofar as it is what each of them is spoken of as being, but in so far as it is of a certain kind and in accordance with its formula.

> Καθόλου δὲ περὶ πάσης αἰσθήσεως δεῖ λαβεῖν ὅτι ἡ μὲν αἴσθησίς ἐστι τὸ δεκτικὸν τῶν αἰσθητῶν εἰδῶν ἄνευ τῆς ὕλης, οἷον ὁ κηρὸς τοῦ δακτυλίου ἄνευ τοῦ σιδήρου καὶ τοῦ χρυσοῦ δέχεται τὸ σημεῖον, λαμβάνει δὲ τὸ χρυσοῦν ἢ τὸ χαλκοῦν σημεῖον, ἀλλ᾽ οὐχ ᾗ χρυσὸς ἢ χαλκός: ὁμοίως δὲ καὶ ἡ αἴσθησις ἑκάστου ὑπὸ τοῦ ἔχοντος χρῶμα ἢ χυμὸν ἢ ψόφον πάσχει, ἀλλ᾽ οὐχ ᾗ ἕκαστον ἐκείνων λέγεται, ἀλλ᾽ ᾗ τοιονδί, καὶ κατὰ τὸν λόγον.[398]

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396 *DA.,* III, 8, 431b26-432a1.

397 Although Aristotle talks about, here, the objects of knowledge and perception, knowing is a kind of thinking, and 'thinking' and 'knowing' are often used interchangeably in *De Anima*, III. I revert to this point later in this section.

398 *DA.,* II, 12, 424a17-24.

Intellect receives the forms of its objects in the same way:[399]

> ...the intellect is in a way potentially the objects of thought, but it is actually nothing before it thinks; potentially in the same way as characters is said to be on a tablet on which nothing is actually written.
>
> …δυνάμει πώς ἐστι τὰ νοητὰ ὁ νοῦς, ἀλλ᾽ ἐντελεχείᾳ οὐδέν, πρὶν ἂν νοῇ: δυνάμει δ᾽ οὕτως ὥσπερ ἐν γραμματείῳ ᾧ μηθὲν ἐνυπάρχει ἐντελεχείᾳ γεγραμμένον.[400]

Those two passages are meant to explain that the objects of perception and thought are forms, not things themselves. That is to say, when we perceive or think an object, we receive only the form of the object without its matter, in the same way that wax receives the impression of a certain matter without receiving the matter itself.

A difficulty in explaining thinking in terms of the reception of forms is that the recipient of forms undergoes no physical change.[401] In the case of thinking (insofar as it

[399] There are parallels between Aristotle's account of thinking and his account of perception. Thinking and perceiving have common features in that (i) both are the process of receiving forms of their objects and (ii) when they are in actuality, i.e. when the intellect thinks or the sense perceives, they become identical with their objects. For similarities between thought and perception, see *DA.,* III, 4, 429a13-429a18.

[400] *DA.,* III, 4, 429b30-430a2.

[401] In the case of perception, this is controversial. For instance, when we see a green object, our eyes do not turn green in the way the physical object is colored. Further, if the reception of a color accompanies physical change in the organ of sight, it is difficult to see how we can see two contrary colors like black and white at the same time; two contraries cannot be in the same place at the same time for Aristotle. However, some commentators argue that Aristotle is speaking only

is the reception of forms), no physical organ of receptivity corresponds to the intellect; if the intellect is not physical, there is no way for the intellect to undergo physical change.[402] In that sense, it can be said that the cognitive reception of a form differs from its physical reception, in that recipient of the form is not physically affected by the form. But if a form is transmitted from an object to a cognitive agent, there must be a matter in which the form is actualized; it is a basic tenet of Aristotle's *hylo-morphism* that a form cannot exist by itself; it is always with its matter. And since when the matter receives the form, the form will act on the matter, this matter must be affected by the form. But if there is no physical change accompanying the reception of a cognitive form, it is hard to see what the matter is in which the form is actualized—especially given that the form was originally embodied in physical matter.

The above passage provides no satisfactory answer to the question of how a received cognitive form is actualized in the intellect or sense. Although Aristotle says that the reception of form takes place without matter, this assertion does not explicate the unique feature of the cognitive mode of receiving forms; not only the intellect but also

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of a physiological change in the sense organ in his claim that perception is the reception of form without matter. For instance, Sorabji thinks that the eye-jelly takes on perceived objects' color (see Sorabji (1993) and (1992)). In fact, the text is not clear on this matter. Aristotle on the one hand maintains that the intellect and sense are not affected by their object, but on the other hand seeks to distinguish the way in which the sense and intellect are unaffected. He says "that the faculties of perception and intellect are not alike in their unaffectedness is clear from [the reference to] the sense-organs and the sense…for the faculty of perception is not independent of the body, whereas the intellect is separable…(…ὅτι δ᾽ οὐχ ὁμοία ἡ ἀπάθεια τοῦ αἰσθητικοῦ καὶ τοῦ νοητικοῦ, φανερὸν ἐπὶ τῶν αἰσθητηρίων καὶ τῆς αἰσθήσεως… τὸ μὲν γαρ αἰσθητικὸν οὐκ ἄνευ σώματος, ὁ δὲ χωριστός…(*DA*., III, 4, 429a29-b5))."

[402] See *Ibid.*, 429a13; 429a29.

physical matter receive only forms without the matter of forms. What we are missing is the explanation of how a form in some physical matter can be transferred to, and actualized in, the intellect without involving another physical matter. The example of wax receiving the imprint of the ring without iron or gold only illustrates that a form in some physical medium is transferred to another physical medium its original matter.

There have been attempts to explain the cognitive reception of forms based on the language of the passage quoted above (*DA.,* III, 12, 424a17- 24). Notably, Alexander of Aphrodisias started commentary on this topic off into a new direction. Seeing that 'τῆς ὕλης' in the first sentence of the passage, 'ἡ μὲν αἴσθησίς ἐστι τὸ δεκτικὸν τῶν αἰσθητῶν εἰδῶν ἄνευ τῆς ὕλης,' can be read to mean either the matter of sensible forms, or the matter of the subject of perception, i.e. the matter of sense organs, Alexander adopted the second reading, to interpret Aristotle as saying that the subject of perception receives forms without its matter[403] That is, in a manner different from the physical reception of forms, the sense receives sensible forms without its matter.[404] This interpretation provides an answer to the question of what the matter is in which a form is actualized: there is no such matter in the sense because the sense receives forms without its matter. Since the sense as the recipient of sensible forms does not have physical matter, the subject of perceiving will avoid having to undergo physical change. The explanatory merit of Alexander's view led to its attracting a host of followers through late

[403] Owen (1981) p. 86.

[404] Alexander of Aphrodisias, *De Anima*, 60.3-62.1. See Alexander (1979) pp. 70-73.

antiquity.[405] Although there were variations among the detail of their interpretations, they had in common a focus on the cognitive subject's immaterial reception of forms.

What they did not deal with, though, was the question of the ontological status of the forms actualized in the sense or intellect. If the intellect receives a form, the form must exist in the intellect. However, for Aristotle, a form cannot exist by itself apart from its matter. If the intellect does not have any matter for receiving a form, how, then, a form can exist in the intellect? The medieval era began to answer these questions by formulating an account of the ontology of immaterially received forms. For instance, according to Thomas, there are two different modes of existence: natural and intentional existence; the sensible forms exist in sensible objects as natural existents and in the intellect or sense as intentional existents.[406] The difference between the two modes of existence lies in the way their forms are actualized; while the forms of natural existence are actualized in physical matter, those of intentional existence are actualized immaterially or without matter. This postulate of things having two modes of existence further explains the identity of the subject of cognition and its objects. The subject and the object are different in natural existence but the same in intentional existence.[407]

I will not now raise the new issue whether the traditional interpretation developed from the ancient through the medieval period correctly represents Aristotle's original position. My interest rather lies with whether that line of interpretation generally

[405] For the Greek commentaries on the immateriality of the reception of forms, see *Ibid.,* p.86-91.
[406] Aquinas (2007) BK. II, Ch. 24, §§ 551-553, pp. 282-283.
[407] Owen (1981) p. 92.

supports Hintikka's view. If so, this would provide grounds for considering Hintikka afresh.

On preliminary consideration, Aquinas' interpretation seems to accord with Hintikka's view, which can be summarized in two claims:

(1) Actualization in the mind is as good as actualization in any other matter.

(2) Anything thinkable is actualizable.

Since Aristotle regards being in actuality as a sufficient condition of being in existence, from (1) and (2), we can infer that

(3) Anything thinkable exists when it is thought.

For Hintikka, the implication of (3) with regard to the ontological status of mathematical objects is obvious: Since mathematical objects are thinkable,

(4) Mathematical objects exist.

Thomas would agree with (1), (2), and (3). However, if by 'thinking' Hintikka merely means 'conceiving', Thomas would not necessarily agree that (4) follows directly from (1), (2), and (3). Hintikka and the longstanding ancient-medieval tradition would construe 'thinking' (νοεῖν) in the relevant Aristotelian texts in different ways. While Hintikka regards anything logically possible as something thinkable, Thomas understands 'thinking' more narrowly, i.e., he reads 'thinking (νοεῖν)' as meaning 'having a form of knowledge (ἐπιστήμ).' Since they use the term differently, the range of the objects of thinking will in turn differ between them. As we will see, while, for Hintikka, a mathematical object can be an object of thought even if it does not exist in the external world, for Thomas, a mathematical object is thinkable only when it does so exist.

Which reading, then, better squares with Aristotle's own view? Let us consider what Aristotle means by 'thinking' in positing identity between thought and its objects; this is the claim on whose ground both Hintikka and Thomas infer the existent ontological status of the objects of thought. 'νοεῖν' is a verb derived from the noun, 'νοῦς (intellect).' Thus, literally 'νοεῖν' means the exercise or functioning of 'νοῦς.'[408] In a broad sense, 'νοεῖν' can mean any intellectual grasp of a universal aspect of a thing or a state of affairs, in contrast with perception, which mostly concerns itself with particulars.[409] More specifically, 'thinking' denotes grasping or intuiting the non-

[408] "Then, that part of the soul which called intellect (and I speak of as intellect that by which the soul thinks and supposes) is actually none of existing things before it thinks (ὁ ἄρα καλούμενος τῆς ψυχῆς νοῦς (λέγω δὲ νοῦν ᾧ διανοεῖται καὶ ὑπολαμβάνει ἡ ψυχή) οὐθέν ἐστιν ἐνεργείᾳ τῶν ὄντων πρὶν νοεῖν (*DA.*, III, 429a22-24))."

[409] "For instance, if we saw the glass to be perforated and the light passing through it, it would also be evident why it burns; when we see in each case separately, we will think at the same time that it is so in every case (οἷον εἰ τὴν ὕαλον τετρυπημέν ἑωρῶμεν καὶ τὸ φῶς διόν, δηλον ἂν ἦν καὶ διὰ τί καίει, τῶ ὁραν μὲν χωρὶς ἐφ'ἑκάστης, νοῆσαι δ'ἅμα ὅτι ἐπὶ πασῶν οὕτωϛ (*Post. An.,* I, 31, 88a15-17))."

demonstrable first principles (ἀρχή) from which demonstrations begin.[410] While Aristotle uses the terminology, 'νοεῖν' in a few different senses, 'νοεῖν X' is always distinguishable from the mere 'conceiving of X' throughout his work, in that it denotes a means of acquiring information of X, which the perception of X alone cannot cover; in that sense, νοεῖν X is to have a kind of knowledge of X.[411] Putting it more generally, for Aristotle, 'νοεῖν' is, along with perception, an epistemic route to access to the external world, in particular, its formal structure.

Such a distinction between νοεῖν and mere conceiving is maintained through *De Anima,* III, in which the identity between thought and its objects is claimed. νοεῖν X is there described as receiving the form of X; since νοεῖν X is nothing but grasping the form of x, νοεῖν X again can be understood as having a certain kind of knowledge of X. In fact, we see, throughout *De Anima*, III, 'thought (νόησις)' and 'knowledge (ἐπι΄στήμ)' are used interchangeably without sharp distinction.[412]

Given that the reception of a certain kind of forms, i.e., intelligible forms, of things in the external world is thus an essential part of the process of thinking, the range of the objects of νοεῖν cannot be broader than the totality of all existing things in the

[410] *Ibid.,* II, 19, 100b5-17. For the necessity of a given first principle in demonstration, see *Ibid.*, I, 3.

[411] Irwin and Fine also agree on this point, see Irwin (1995) p. 619.

[412] It is interesting that Hintikka also uses the term 'thinking' without distinguishing it from 'knowing'. In claiming, though, that thinking is as good an actualization as physical actualization, Hintikka's term 'thinking' designates nothing more than 'conceiving.' His argument for mental actualization relies heavily on this ambiguous usage of 'thinking.' See Hintikka (1966), p. 208.

external world. Aristotle confirms this by saying that "thinking and understanding are thought to be like a form of perceiving for in both of these the soul judges and recognizes some *existing* thing (δοκεῖ δὲ καὶ τὸ νοεῖν καὶ τὸ φρονεῖν ὥσπερ αἰσθάνεσθαί τι εἶναι ἐν ἀμφοτέροις γὰρ τούτοις κρίνει τι ἡ ψυχὴ καὶ γνωρίζει τῶν ὄντων)."[413] Thus, when he says that the intellect thinks all things,[414] by all things he means all existing things, not all logically conceivable things or all possible postulates.[415] In this sense, we can say that Hintikka's identification of thinkability with logical possibility is a mistake arising from a misunderstanding of Aristotle's usage of the term, 'νοεῖν'.[416]

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[413] *DA.,* III, 3, 427a19-21.

[414] *Ibid.*, 3, 429a18.

[415] "…the soul is in a way all *existing* things; for [existing things] are either objects of perception or objects of thought (…ἡ ψυχὴ τὰ ὄντα πώς ἐστι πάντα: ἢ γὰρ αἰσθητὰ τὰ ὄντα ἢ νοητά (*Ibid*., 8, 431b21-22)).

[416] One might object that Aristotle uses 'νοεῖν' in a much broader sense even within *De Anima*, III. For instance, Aristotle says that "Nor again is thinking, which can be right or wrong; right thinking [includes] understanding, knowledge, and true belief, wrong the opposite of these (ἀλλ' οὐδὲ τὸ νοεῖν, ἐν ᾧ ἐστι τὸ ὀρθῶς καὶ τὸ μὴ ὀρθῶς, τὸ μὲν ὀρθῶς φρόνησις καὶ ἐπιστήμη καὶ δόξα ἀληθής, τὸ δὲ μὴ ὀρθῶς τἀναντία τούτων) (*Ibid.,* 427b8-11)." In this passage, however, Aristotle divides thinking into two sorts: right thinking and wrong thinking. We might identify the former, right thinking, with the type of thinking so far we have discussed, namely, thinking as the receiving or actualization of a form including knowledge of that form and its object. But since wrong thinking includes false belief, it might be said that there is room for non-beings or fictional entities in thinking in an Aristotelian sense; Aristotle elsewhere maintains that a statement predicated of a non-being is always false (see *Cat.,* 10, 13b11-35). However, thinking in the second sense has nothing to do with actualization in the intellect. Mental actualization is the reception of form without matter, whilst the intellect cannot receive a form from any fictional entity.

Since the forms actualized in the intellect are not created but only received therein,[417] for a form, *F*, to be actualized in the intellect, there must be already something which actualizes or embodies *F* outside the intellect. Thus, even if it is accepted that anything thinkable can be actualizable, and being in actuality in the intellect can be seen as a way of existence, the claim that anything conceived by the intellect can be said to exist cannot be properly Aristotelian. For Aristotle, thinking is distinguished from mere conceiving; and something can be actualized in the intellect only if that thing is already actualized in the external world. Therefore if there is no actuality of mathematical objects outside the intellect, they cannot be actualized in the intellect, either.[418] Moreover, what is merely potential cannot be said in Aristotle's metaphysics to *exist.*

[417] For Aristotle, forms are not generated but given and eternal. See *Met.,* VII, 8, 1033b1-20; VIII, 3, 1043b15-24; XII, 3, 1069b35-1071a4.

[418] Thus, we must distinguish the mental construction of geometrical constructions by the intellect from forms' actualization in the intellect in general, even though Aristotle calls the former 'actualization' as well as the latter (see *Met*., X, 1, 1051a21).

Appendix

Fictionalism in Aristotle's Theory of Number

1. Number as a Property of Plural Things

Whereas realism proved untenable for Aristotle in the case of geometry, most commentators have thought the philosopher can hold to a consistent realistic theory of number. One of Aristotle's main claims about number is that a number is a property of plural things. This has led commentators to try to show that Aristotle holds a plausible realistic view of number, since on their view the idea that a number is a property of plural things may be adequately defended.

In my view, however, this is highly misleading: First, it presumes an incorrect relationship of priority between the branches of mathematics for Aristotle, given that Aristotle founded arithmetic on the basis of geometry. Secondly, it comes into conflict with Aristotle's claim that mathematical objects do not exist in actuality; we do not have any ground for believing that he confines this claim only to geometrical objects. More importantly, Aristotle distinguishes arithmetical number from sensible number; while the latter is composed of sensible unities such as cows or men, the former is composed of the units which are "absolutely indivisible (πάντῃ ἀδιαίρετος)." Further, he argues that the unity of arithmetical number "is most exact and in all other cases we imitates this measure (ἀκριβεστατον......ἐν δὲ τοῖς ἄλλοις μιμοῦνται τὸ τοιοῦτον)."[419]

Differently from most modern mathematicians, Aristotle does not seem to think that there is any fundamental difference between geometry and arithmetic;[420] for Aristotle, both sciences are the study of quantity: the former continuous quantity, the latter discrete.[421] Thus, as he does geometrical objects, he explains arithmetical objects, or numbers, in terms of abstraction: Arithmetic is a science which studies sensible objects *qua* indivisible[422] For instance, while the geometer studies a man *qua* a solid or figure, the arithmetician studies a man *qua* man; a man *qua* man is indivisible in the sense that any divided corpse would no longer be a man.

We have seen before, that when X studies *a qua* F, X studies the *kath'hauta* properties of F which belong to *a,* and ignores other properties of *a*. Namely, X studies the properties which belong to *a* by virtue of *a*'s being F. Now, since arithmetic studies things *qua* 'indivisible', it must study those properties of things which belong to them by virtue of those things' being indivisibile. But since the subject of arithmetic is number, properties of number are by that token held to be properties of things *qua* indivisibles. But what properties of indivisible things could these numerical properties be? To answer this question, first, we should consider Aristotle's definition of number.

[419] *Met.*, X, 1, 1053a1-2. Aristotle here means unity by 'measure' or unit. See Appedix. 1, n.402. I discuss Aristotle's distinction between two different kinds of number in Appendix, 3.
[420] For the opposite position, see Frege (1968) §13.
[421] See *Cat.*, 8, 4b20-31; *Met.*, XIII, 9, 1085b15-22; 2, 1077b17-22; 3, 1078a2-3; 9, 1085a20.
[422] *Ibid.*, 3, 1078a1-3.

Aristotle defines number in several ways: as a 'plurality of unities (ἕνα πλείω),'[423] as 'the plurality of measures (τὸ πλῆθος μέτρων)' [424] or as 'the plurality of indivisibles (τὸ πλῆθος ἀδιαιρέτων).'[425] The concepts of unity (ἕν), measure (μέτρον) and indivisible (ἀδιαιρέτος) are closely connected with each other.[426] Aristotle distinguishes different senses of unity: the whole, the individual, and the universal. All these, he says, are unities because they are *indivisible* in some respect—some in terms of quantity, and others in concept or formula.[427] Now, in this broad sense of unity, a measure may also figure as a unity, in that it is indivisible according to quantity: When we use 'a foot' as the measure of length, a foot is indivisible in respect of the length it captures.[428] Likewise, when we count the number of men in a group, 'a man' is the measure by which the quantity of the group is known; i.e., to know the quantity of a group of men is to know how many men there are in the group. Thus, in fact, the three definitions of number all describe essentially the same idea.

According to Aristotle's definitions of number, number is a number of unities; and these unities are always certain measures such as a man, a foot, a red flower, etc.[429] But since the unities of a number are plural things conceived of as indivisible; Aristotle

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423 *Phys.*, III, 7, 207b7.
424 *Met.*, XIV, 1, 1088a6.
425 *Ibid.*, XIII, 9, 1085b23.
426 In fact, Aristotle uses 'unity' interchangeably with 'measure' in his talk of number; e. g., he says that 'that unity denotes a measure is obvious.' (*Met.,* XIV, 1, 1087b34-5).
427 *Phys.*, III, 7, 207b6-10; *Met.*, X, 1, 1052a24-34; XIII, 7, 1082a20-23; XIV, 1, 1087b33-36.
428 *Ibid.*, X, 1, 1053a20-22; XIV, 1, 1088a7-10.
429 *Ibid.*, XIV, 1, 1088a10-14. In this way Aristotle says that 'a number is always a number of something (*Ibid.*, XVI, 5, 1092b20).'

also claims that number is a *property* of plural things. This implies that the existence of number depends on the existence of indivisible things, since there can be 'plurality' only when there are plural things. Thus, Aristotle says that number is derivative, and argues that the substantive usage of number derives from its adjectival usage.[430] However, his idea that number is a property of a group of things needs to be able to confront some serious objections..

2. Aristotle's Nominalism and the Truth of Arithmetic

First of all, there seems to be a crucial difference between numerical and other properties: Numerical properties are not distributive. For example, consider:

(1) Solon, Socrates, Plato are wise.

and

(2) Solon is wise.

We can derive (2) from (1). However, in the case of

(3) Solon, Socrates, and Plato are three.

and

[430] *Phys.*, III, 7, 207b8-10. This is sharply in contrast with the Platonic view; for Plato, number is self-subsistent and causally prior to sensible things.

(4) Solon is three.

(3) does not imply (4). From this non-distributivity of numerical properties, Frege infers that number is not a property of physical objects.[431] Equally, though, many other properties besides numerical are also non-distributive. For instance, even if a team is well-organized, it does not follow that each member of the team will be well-organized. Thus, the non-distributivity of numerical properties only shows that numbers are used as adjectives not of singular but of plural terms.[432]

So far, we have seen that: (i) a number is a property of plural unities; (ii) numerical properties are not distributive; i.e., the subject of a numerical adjective is plural.[433] Keeping these points in mind, we can now move to formulate Aristotle's concept of number.[434] To begin with, let us assume that variables x, y, and z range over sensible substances, and F, G, H stand for distributive or non-distributive properties. In addition, for a non-distributive property, let us introduce a sign for plural terms as distinguished from singular terms: [x] stands for a group of x's, and G[x] means that a group of x's is G.[435] Since we accept that a number is a property of plural unities, 'two

[431] Frege (1968), §22.

[432] For the distinction between a distributive and a non-distributive property, see Bell (1990) pp.68-69

[433] This explanation of number excludes 0 and 1 from numbers. I discuss this matter in Appendix, 3.

[434] There have been similar attempts to formulate the idea that number is adjectival. Among these, the most elaborate and complete is that of Bostock (1974).

[435] An important difference between '[ ]' and '{ }' is that :$\{\{x, y\}, \{z\}\} \neq \{x, y, z\}$, whereas $[[x, y], [z]] = [x, y, z]$. I owe this symbolism to Bell. See Bell (1990) pp. 64-69.

Fx's' can be expressed as 2[x: x is F]. Here, F is a property which satisfies one of the conditions of unity mentioned above; e.g., 'being a tree' can be an instance of F, but 'being water' cannot, because x *qua* water yields no concept of the indivisible unity of x.[436] Now, given the first-order predicate calculus with non-identity, it is a simple matter to express 'there are exactly two Fx's':

$$2[x: x \text{ is } F] \text{ iff } \exists x \exists y \forall z(Fx \& Fy \& x \neq y \& (Fz \rightarrow z=x \vee z=y))$$

More generally, we can formulate n[x: x is F] as follows:

$$n[x: x \text{ is } F] \text{ iff } \exists x1 \ldots \exists xn \forall y(Fx1 \& \ldots Fxn \& x1 \neq x2 \& x1 \neq x3 \& \ldots \&$$

$$x1 \neq xn \& x2 \neq x3 \& \ldots x2 \neq xn \& \ldots \& xn\text{-}1 \neq xn \& (Fy \rightarrow (y=x1 \vee \ldots \vee y=xn)))$$

One problem with this formula is that it forces us to accept non-identity in the absence of any principle of differentiation for x and y *qua* F. Although x and y are the same kind of unity in virtue of their being F, this property does not differentiate x from y; namely, we have no ground for saying that Fx≠Fy given that the two terms are the same

[436] Here, F is to be a property such that for any x, x *qua* F is indivisible. For instance, 'being a man' can be an instance of F, since if a man is divided into pieces, each piece is no longer a man; thus, 'being water' cannot be an instance of F, since any part of the whole which is water is water. Mignucci argues that Fx is a unity if and only if F is a *sortal* concept or predicate. See Mignucci (1991) p. 191. For more discussion of the notion of a sortal concept, see Strawson (1959); Wiggins (1980); Grandy (2008).

qua F. But it should be noted that variables x, y, and z range over sensible substances; and, for Aristotle, the numerical difference of substances is given primitively.[437] Further, in his ontology, beings belonging to the other categories are ontologically dependent on substances, so that F is always a property of a particular substance. Moreover, by virtue of F's being a property of an individuated substance, Fx is also individuated. Similarly, F[x] can be differentiated from F[y], not by the fact that [x] is F and [y] is F, but because [x] is different from [y]; and [x] and [y] are different because the members of [x] and members of [y] are different.

However, is to know that $\exists x \exists y \forall z(x \neq y \& (z=x \vee z=y))$ really to know that there are *two* things? Do we not rather only know that x is different from y?[438] Putting it another way, can the concept of number be simply reduced to concepts of unity and difference? Aristotle's definition of number seems to support such a reduction. One of his definitions of number is that number is the plurality of unities,[439] and to know that there are plural unities, we only have to know that there are unities which are distinct from each other.

[437] Concerning the problem of individuation in Aristotle, interpretations divide into three main groups: The first argues that matter is the 'individuator' of things (see Lukasiewicz, Anscome, & Popper (1953); Lloyd (1970)). But according to the second interpretation, form, rather, 'individuates' (see Sellars (1957) and Albritton (1957)). On the third view, neither form nor matter individuates things, but form or matter are themselves individuated by the primitively given individuality in a particular: *this* form is differentiated from *that* form by the fact that *this* form is the form of *this* thing. For the third line of interpretation, see Charlton (1972) and Regis (1976). However, all lines agree that Aristotelian substances can be individuated. Since this is not the place to discuss this topic, my discussion adopts the third interpretation without further argument.

[438] Mignucci (1991) p. 195.

[439] For an argument for this claim, see Mignucci (1991) pp.194-196.

But another difficulty follows: In the formula, variables range over individual substances, but we ascribe number not only to groups of substances, but also to other categories of beings, including plural references. For instance:

(1) There are two cows.

(2) There are two dogs.

(3) There are two colors.

(4) There are two pairs of shoes.

If we accept all these sentences are meaningful, and the sentences (1) through (4) are true, then, the sentences below must also be true:

(5) The number of cows = the number of dogs.

(6) The number of cows = the number of colors.

(7) The number of cows = the number of pairs of shoes

But it is not easy to find a commonly shared property among two cows, two dogs, two colors and two pairs of shoes. Suppose that [x] and [y] are two groups of unities and that y and x are heterogeneous. It seems obvious that, although a non-distributive property of a group of things cannot be ascribed to each member of the group, nevertheless, the non-distributive properties a group can have is determined by what properties each member of the group has in some sense individually.[440] Therefore, if a number is a non-distributive attribute of a group of unities, and x is entirely heterogeneous with y, then [x] and [y] should have different non-distributive properties; e.g., it is unlikely that there is a common property between a group of trees and a group of colors. The following passage of Aristotle speaks to just this problem:

> The measure must always be some one and the same thing applying to all cases, e.g. if the things are horses, the measure is horse, and if they are men, man. If they are a man, a horse, and a god, the measure is probably living things, and the number of them will be a number of living things. If the things are man and white and walking, there will be scarcely a number of these, because they all can belong to numerically one and the same thing; nevertheless, the number of these will be a number of classes, or some other equivalent term.
>
> δεῖ δὲ ἀεὶ τὸ αὐτό τι ὑπάρχειν πᾶσι τὸ μέτρον, οἷον εἰ ἵπποι, τὸ μέτρον ἵππος, καὶ εἰ ἄνθρωποι, ἄνθρωπος. εἰ δ᾽ ἄνθρωπος καὶ ἵππος καὶ θεός, ζῷον ἴσως, καὶ ὁ ἀριθμὸς αὐτῶν ἔσται ζῷα. εἰ δ᾽ ἄνθρωπος καὶ λευκὸν καὶ βαδίζον, ἥκιστα μὲν ἀριθμὸς τούτων διὰ τὸ ταὐτῷ πάντα ὑπάρχειν καὶ ἑνὶ κατὰ ἀριθμόν, ὅμως δὲ γενῶν ἔσται ὁ ἀριθμὸς ὁ τούτων, ἤ τινος ἄλλης τοιαύτης προσηγορίας.[441]

The idea is that, insofar as we find a common measure for a group of things, we can *number* the group of things. This idea may be applied to explain the truth of the sentences above: In the case of (5), we may say that what is the measure or unit is neither cow nor dog, but animal or living thing, so that the same number, namely, the number of animals,

[440] Mignucci (1991).
[441] *Met.*, XIII, 5, 1088a8-14.

can be predicated of each of the groups. And the number is a common formal aspect of both groups of unities, and unities play the role of that matter which receives number as its form.[442] That is, both the group of dogs and the group of cows share a certain formal property which satisfies the formula:

$$\exists x \exists y \forall z(Fx \& Fy \& x \neq y \& (Fz \rightarrow z=x \vee z=y))$$

And, in the case of (6), color and cow could be thought of as unities in the sense that both are indivisible in some respect; and we might even suppose that both (2) and (3) satisfy the formula above, even allowing the variables, x, y, and z to range over beings in all categories.

However, in the case of (7), there is some difficulties in explaining how (4) satisfies such a formula; since, in (4), the unity is 'pair of shoes', if x, y, and z are arguments for unities, variables should range over plural terms. More importantly, in the case of (4), what number can be assigned to the group of shoes depends entirely on the way in which we carve them out: two pairs of shoes may equally be picked out according to a different measure of unity as four shoes. Thus, even if we expand the scope of variables that may be substituted for plural terms, whether the proper logical form of (4) is

[442] Aristotle assimilates unity and number to matter and form, respectively. See *Met.*, XIII, 8, 1084b2-6.

$\exists x \exists y \forall z(Fx \& Fy \& x \neq y \& (Fz \rightarrow z=x V z=y))$

or

$\exists x1 \exists x2 \exists x3 \exists x4 \forall x5((Fx1 \& Fx2 \& Fx3 \& Fx4 \& x1 \neq x2 \& x1 \neq x3 \& x1 \neq x4 \& x2 \neq x3$

$\& x2 \neq x4 \& x3 \neq x4 \& (Fx5 \rightarrow (x5=x1 Vzx5=x2 V x5=x3 V x5=x4)))$

would seem to depend on our conceptualization.

A possible realist interpretation would be that, for Aristotle, only a substance is a unity in its own right; it has its own essence whereby its other properties are causally unified; other senses of unity are just analogous or derivative, so that the number of *substances* counts as the only genuine number. But we can accept this proposal only at the expense of abandoning explaining the truth of sentences which describe other categories of beings in terms of number, such as (3), (4), (6) and (7). In addition, we should remember that, for Aristotle, mathematical objects exist as matter (ὑλικῶς) in sensibles.[443] But it seems that the number of sensible unities does not exist as matter, but, rather, number is a certain form of those unities. Furthermore, although for Aristotle the paradigm case of unity is a substance, he identifies a primary sense of unity under the category of quantity.[444]

3. Sensible Number and Arithmetical Number

[443] *Met.*, XIII, 3, 1078a28-30.

[444] *Ibid.*, X, 1, 1052b15-20. Also see Cleary (1995) p. 376.

In this case, it seems that Aristotle has to confront a dilemma: If unity is so universal as to be applicable to anything numerable, then his thesis that a number is a property of plural unities cannot be maintained; since there is no such attribute common to every equinumerous group. On the other hand, however, if the usage of 'unity' is restricted to substances, we cannot explain the obvious truth of some numerical statements. At this point, Aristotle seizes the first horn of the dilemma. His strategy to preserve the objectivity of arithmetic is to make a distinction between two kinds of numbers: mathematical number and sensible number. To begin with, let us first consider the following passage:

> …the number of the sheep and of the dogs is the same number if each [number] is equal, *but the ten is not the same [ten] nor are there ten of the same [kind]*…For a thing is called the same [kind] if it does not differ by the differentia [of that kind], …Therefore it is the same number (for their number does not differ by the differentia of number), but *it is not the same ten; for the things of which it is said differ; one is of dogs, and the other of horses.*
>
> …ἀριθμὸς μὲν ὁ αὐτὸς ὁ τῶν προβάτων καὶ τῶν κυνῶν, εἰ ἴσος ἑκάτερος, δεκὰς δὲ οὐχ ἡ αὐτὴ οὐδὲ δέκα τὰ αὐτά, ...ταὐτὸ γὰρ λέγεται οὗ μὴ διαφέρει διαφορᾷ...καὶ ἀριθμὸς δὴ ὁ αὐτός (οὐ γὰρ διαφέρει ἀριθμοῦ διαφορᾷ ὁ ἀριθμὸς αὐτῶν), δεκὰς δ᾽ οὐχ ἡ αὐτή: ἐφ᾽ ὧν γὰρ λέγεται, διαφέρει: τὰ μὲν γὰρ κύνες, τὰ δ᾽ ἵπποι.[445]

This passage is somewhat puzzling in apparently offering two contradictory assertions: (i) The number of ten dogs and the number of ten sheep is the same number, and yet (ii) they

[445] *Phys.*, IV, 14, 224a2-15.

are not the same ten or decad. To take the latter claim first, Aristotle appears to posit that, if two groups of unities are qualitatively different, then their numbers are also different. Nevertheless, he maintains that two qualitatively different groups can have the same number. This seemingly contradictory claim is based on his distinction between two different kinds of numbers: the number which we count and the number we count with. He says that:

> Time then is a kind of number. But n*umber is in two ways (for we call number both what is counted or countable and that by which we count).* Time, then, is what is counted, not that by which we count. That by which we count is different form that which is counted.
>
> ἀριθμὸς ἄρα τις ὁ χρόνος. ἐπεὶ δ' ἀριθμός ἐστι διχῶς (καὶ γὰρ τὸ ἀριθμούμενον καὶ τὸ ἀριθμητὸν ἀριθμὸν λέγομεν, καὶ ᾧ ἀριθμοῦμεν), ὁ δὴ χρόνος ἐστὶν τὸ ἀριθμούμενον καὶ οὐχ ᾧ ἀριθμοῦμεν. ἔστι δ' ἕτερον ᾧ ἀριθμοῦμεν καὶ τὸ ἀριθμούμενον.[446]

For Aristotle, time is a kind of number in the sense that we measure motion by time in respect of before and after, just as we measure things by number in respect to more and less. And unities of time are discrete sequences of the temporal continuum divided by 'nows.' But, since the temporal continuum is after all identified with the movement of heavenly bodies,[447] time can be seen as an example of numbers whose units are concrete. The important point here is that Aristotle marks a distinction between the number with

[446] *Ibid.*, IV, 12, 219b5-9. Also see 220b8-12.
[447] *DC.*, II, 11, 291b3-16. On the conceptual relationship between time and number in Aristotle, see Annas (1975).

which we count and the number of concrete unities counted. This explains how two qualitatively different groups may have the same number. One way to know the cardinality of a group of things is to correlate one-to-one members of the group to the elements of the number series:[448] i.e., for two qualitatively different unities x and y, the number of [x] and the number of [y] is the same *n* iff each x and each y correlates 1-1 to the members in the number series from the first to the *n*th member, respectively. Since, then, the number of [x] is the result of the act of counting x's by the ordinal number, the existence of the number of [x] is partly mind-dependent.

> One would ask whether if there were no soul, there would be time or not. For if that which counts cannot exist, there cannot be something to be counted, either, so that it is obvious that there cannot be number; for number is either what has been, or what can be, counted. But if nothing but soul, or intellect of soul, is qualified to count, it is impossible for there to be time if there is no soul.
>
> πότερον δὲ μὴ οὔσης ψυχῆς εἴη ἂν ὁ χρόνος ἢ οὔ, ἀπορήσειεν ἄν τις. ἀδυνάτου γὰρ ὄντος εἶναι τοῦ ἀριθμήσοντος ἀδύνατον καὶ ἀριθμητόν τι εἶναι, ὥστε δῆλον ὅτι οὐδ᾽ ἀριθμός. ἀριθμὸς γὰρ ἢ τὸ ἠριθμημένον ἢ τὸ ἀριθμητόν. εἰ δὲ μηδὲν ἄλλο πέφυκεν ἀριθμεῖν ἢ ψυχὴ καὶ ψυχῆς νοῦς, ἀδύνατον εἶναι χρόνον ψυχῆς μὴ οὔσης.[449]

Nevertheless, it seems that the number with which we count does not have to be constituted purely subjectively. On the contrary, counting presupposes knowledge of this counting number. But what is the number with which we count? In fact, Aristotle

[448] Benacerraf (1965) p. 50

[449] *Phys.*, IV, 14, 223a21-26

distinguishes arithmetical number from sensible numbers in several places, suggesting in each that by the counting number he means to designate arithmetical number.[450] This conjecture is once more supported by his explanation of arithmetical number; among others, one passage suggests a procedure of the recursive generation of arithmetical numbers in their proper order.[451] But these facts only show that by the counting number Aristotle means arithmetical number; we do not know yet what arithmetical number is.

Although the texts do not provide any explicit indication, we still have some pointers to help us infer what Aristotle might mean in talking about arithmetical number. First, for Aristotle, a number is always a number of something; he defines number as the plurality of unities. So, it is quite reasonable to think that arithmetical number is also some sort of composition of unities. Secondly, it would appear that these unities cannot be sensible or concrete, because the counting number is distinguished from all numbers appertaining to sensible unities. Further to these pointers, we can find a further clue in the practice of Greek mathematics: in Greek arithmetic, number and its operations are represented in terms of a metrical geometry, in particular by the manipulation of line

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[450] "Now where it is thought impossible to take away or to add, there the measure is exact, so that the measure of number is most exact; for we posit the unit as absolutely indivisible; *and in all other cases we imitate this sort of measure* (ὅπου μὲν οὖν δοκεῖ μὴ εἶναι ἀφελεῖν ἢ προσθεῖναι, τοῦτο ἀκριβὲς τὸ μέτρον διὸ τὸ τοῦ ἀριθμοῦ ἀκριβέστατον: τὴν γὰρ μονάδα τιθέασι πάντῃ ἀδιαίρετον *ἐν δὲ τοῖς ἄλλοις μιμοῦνται τὸ τοιοῦτον* (*Met*., X, 1, 1052b35-1053a2))." Also see *Ibid*., XIII, 8, 1083b16; XIV, 5, 1092b22-25.

[451] He says, "…number must be produced by addition; e.g., 2 by adding 1 to another 1, and 3 by adding another 1 to the 2, and 4 similarly (ἀνάγκη ἀριθμεῖσθαι τὸν ἀριθμὸν κατὰ πρόσθεσιν, οἷον τὴν δυάδα πρὸς τῷ ἑνὶ ἄλλου ἑνὸς προστεθέντος, καὶ τὴν τριάδα ἄλλου ἑνὸς πρὸς τοῖς δυσὶ προστεθέντος, καὶ τὴν τετράδα ὡσαύτως (*Ibid*., XIII, 7, 1081b14-17))."

lengths.[452] Thus, it may be suggested that, for Aristotle, the unities of arithmetical number are nothing other than geometrical line lengths; and that each number, n, is a notation for a line composed of n line lengths. While line length is distinguished from a concrete measure such as a foot or meter in that it has only one property, 'being the same length,' unlike platonic pure unit, it admits of differentiation by virtue of its position on the line.[453] This allows us to avoid the problem of the individuation of units. In addition, we have seen that geometrical objects exist only as matter or in potentiality in sensible objects. Thus, to the extent that geometrical lines exist as matter in sensible substances, arithmetical number exists as matter in sensible substances; the unit of arithmetical number is geometrical line length. We can then say that, like geometrical objects, mathematical numbers also exist as matter in sensible or concrete objects.

Nevertheless, these formulations do not solve every problem. First of all, whether a number is arithmetical or sensible, it is always of plural things. This precludes 1 from membership in the set of numbers. In this instance, it is worth making the point that in the ancient Greek, 1 was not regarded as a number. For instance, Euclid defines a number as

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[452] Gaukroger (1980) pp.190-191; Knorr (1975) p. 173. Aristotle himself also represents numbers or times by line length; see *Phys.,* VI, 7, 237b34-238a1. Many commentators have traditionally reformulated the Greek geometrical proof in terms of algebra: e. g., Heath (1921). But, as Gaukroger points out, this rephrasing distorts the concerns of Greek arithmetic and geometry. Among more recent commentators, Szaboá showed that modern algebraic reformulation trivializes some proofs in the *Elements* (e.g., *Element,* IX, Pro. 8 and 9), although Euclid clearly thought these proofs essential in their own terms; see Szaboá (1978) pp. 189-198. For differences between the concept of modern symbolic number and that of Greek geometrical number, see Klein (1968) and Mahoney (1980).

[453] On the difference between line and line length, see Gaukroger (1980) p. 192 and Gaukroger (1982) p. 319.

"the plurality composed of units" (τὸ ἐκ μονάδων συγκείμενον πλῆθος).[454] An obvious implication of this definition is that a unit is not a number; a single unity cannot compose a plurality. Nevertheless, Euclid sometimes treats 1 as a number, ignoring the distinction between number and unity. Aristotle's treatment of 1 is similar to Euclid's. On the one hand, he argues that a unit is not a number since it is the principle of number.[455] But he also treats 1 in the same way as other numbers in other places.[456] This treatment is not necessarily inconsistent. Since Aristotle treats a number as a composite of unities, unity is always susceptible of combination with other unities. Thus, a unity can be used in arithmetical operations with numbers insofar as a number is nothing but combined unities. Aristotle should perhaps rather have thought that his explanation of number had the merit of providing a philosophical background for his contemporary concept of number.[457]

In a similar vein, we can also question how Aristotle should deal with rational

[454] *The Elements*, VII, Def. 2.

[455] "For one means the measure of some plurality and number means a measured plurality and a plurality of measures. Thus, it is reasonable that one is not a number; for the measure is not measures, but both the measure and the one are principles (σημαίνει γὰρ τὸ ἓν ὅτι μέτρον πλήθους τινός, καὶ ὁ ἀριθμὸς ὅτι πλῆθος μεμετρημένον καὶ πλῆθος μέτρων διὸ καὶ εὐλόγως οὐκ ἔστι τὸ ἓν ἀριθμός: οὐδὲ γὰρ τὸ μέτρον μέτρα, ἀλλ᾽ ἀρχὴ καὶ τὸ μέτρον καὶ τὸ ἕν (*Met.*, XIV, 1, 1088a4-8))"

[456] *Ibid.*, XIII, 6, 1080a30-35; X, 6, 1056b25; XIII, 9, 1085b10; 6, 1080a24; 1080b35; *Phys.*, IV, 12, 220a27-32; III, 6, 206b 30-32; *Cat.*, 6, 5a31

[457] It is interesting that, even in modern English, numerical expression like 'numerous,' 'a number of,' etc., are used with plural terms. The issue as to whether 1 is a number or not came up again in the debate between Frege and Husserl. Frege argues that 0 and 1 should be seen as numbers on the grounds that they are adequate answers to the question, 'How many?' However, Husserl points out that 0 or 1 cannot be regarded as an answer to the question, because they are only negative response to that question, namely, to say that there is 0 or 1 thing amounts to saying that there is *nothing* or *not many* things. For a defense of Husserl's response to Frege, see Bell (1990).

and irrational numbers, in that both types of number fail to satisfy his definition of number. On the geometrical account of number, rational number was regarded as a proportion between numbers, rather than as another kind of number; before Eudoxus' general proportional theory, it was customary to represent the ratio of two numbers by straight lines represented as perpendicular to one another. For instance, 'three by two' or 'thrice two' referred to either a rectangle or a ratio of these quantities.[458] In that sense, we might say that rational numbers are not instances of number for Aristotle. The more problematic case is of irrational number, especially since Aristotle was fully aware of the existence of incommensurables.[459] On the question of incommensurability, an interesting point is that, unlike Plato, Aristotle does not seem to have been much worried by it. Among his remarks on the incommensurable, the most revealing is that some magnitudes are measured not by one, but two different measures.[460] This seems to indicate that he just accepts that there is no measure by which every magnitude can be measured. But this really matters only to those who, like Pythagoras or later Plato, adopt a kind of mathematical reductionistic cosmology suggesting that everything is caused or explained in terms of number.[461] This is an idea Aristotle resists. Thus, as a non-mathematical reductionist,[462] Aristotle can admit the existence of quantity which cannot be expressed

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[458] Heath (1949) pp. 221-223. In *Elements*, Book VII, where Euclid sets out proportional theory, he uses parallel lines, instead.

[459] *APo.,* I. 23. 41a23-7; II. 16. 65a4-9; I. 33. 89a29-30; *Met.*, I. 2. 983a13-20

[460] *Met.*, X, 2, 1053b31-1053a18

[461] Cleary (1995) pp. 100, 131-4, and 141-2.

[462] For Aristotle's anti-mathematical reductionism in physics, see *DC.*, III, in particular, 1, 300a15-19.

by number or ratio of number, without being compelled to consider this under the category of number.

Conclusion

Plato argues that there must exist objects of sciences apart from sensible particulars, based on the differences between them. One such difference is that sensible particulars have properties that the sciences are not concerned with, e.g., a sensible particular triangular thing differs from a geometrical triangle in that the former has non-geometrical properties, besides its necessary properties of triangularity, such as its color, material and weight, etc. However, for Aristotle, the object of a science is to be conceived as a certain aspect of a sensible object, from which it may be conceptually separated by abstraction. For instance, we may 'select out' the circularity of a red bronze circular thing by abstracting or removing other irrelevant properties such as its redness and brazenness. Since such separation through abstraction is only conceptual, not ontological, Aristotle argues, there is no need to posit a geometrical circle apart from sensible circular things.

In my view of Aristotle's abstraction, when a science studies X *qua* Y, it studies only the *kath' hauta* properties of Y among properties of X, and those *kath' hauta* properties can be selected through abstraction. We also saw that Aristotle's abstraction is distinguished from empiricists' abstraction in that it can be described as a linguistic process picking out a certain group of predicates from a subject, rather than an epistemic process of obtaining a universal from particulars. My interpretation is also different from the traditional view. Traditionally Aristotle's abstraction has been understood as a kind of mental activity such as thinking or paying attention or ignoring, etc; for instance, to abstract a property, *F*, from an object, *a*, means conceiving of *a* only as *F* or paying attention only to *a*'s *F-ness* and ignoring *a*'s other properties. I showed that Aristotle's abstraction has nothing to do with such mental activities and that no decisive textual evidence recommends the traditional mentalist line of interpretation.

Since, by Aristotle's theory of abstraction, the objects of the sciences are obtained from sensible particulars by abstraction, mathematical objects have to be properties of sensible particulars if mathematics is to be a science. Aristotle confirms his commitment to this naïve realism in several places. His mathematical naïve realism, however, comes up against the precision problem, namely that most mathematical objects are not perfectly instantiated by sensible particulars. Aware of this problem, Aristotle claims that mathematical objects are not in sensible objects as such but exist only as matter. This claim, though, raises two further issues: one of compatibility with his naïve realism, and another of the sense of 'matter' it invokes.

On the first issue, I argued that the claim cannot be made consistent with his mathematical naïve realism, hypothesizing that Aristotle's awareness of the precision problem compelled him to abandon his original naïve realism and later develop a second, distinct and incompatible theory. On the second issue, we examined four interpretations; my evaluation of each focused on its possible consistency with Aristotle's scientific realism.

We began with Lear's view. According to Lear, what we abstract from sensible objects are only basic elements of geometrical figures, such as points, lines, and circles; a geometrical figure is constructed out of these elements. While sidestepping the precision problem, Lear's interpretation runs into, among others, the problem that it turns mathematical objects into fictional entities in Aristotle's ontology. Although Lear insists that a geometrical figure constructed out of elements abstracted from sensible objects has a link with reality, we saw that mathematical fictionalism is not compatible with Aristotle's scientific realism, and that there is no way to explain mathematical truth in terms of Aristotle's own theory of truth if mathematical objects are fictional.

The second view we considered interprets the matter of mathematical objects as pure extension. According to this view, we obtain mathematical matter from sensible objects by abstraction and construct geometrical figures by imposing their forms on this 'matter'. Besides the difficulty of accounting how we can obtain Euclidean space by abstraction from the sensible world, this view, like Lear's, made mathematical objects fictional. Although the matter of a figure ultimately derives from sensible objects, this alone does not guarantee the reality of mentally constructed geometrical figures.

Since most geometrical objects do not exist in actuality, we tried to attest to their existence by identifying their matter with some sort of potential being; Aristotle certainly regarded some kinds of potentiality as modes of existence. But we soon encountered difficulties in assimilating the matter of geometrical objects to potentiality understood as another mode of existence: according to Aristotle's actualism, actuality is prior to potentiality in existence; so if geometrical objects do not exist in actuality, they cannot exist in potentiality, either. There is, though, one exception to Aristotle's actualism: the potentiality of artifacts, which Aristotle identifies with their matter, is prior to actual artifacts in existence.[463] However, this did not seem to provide a ground for regarding mathematical objects as being existent, since, unlike the potentiality of artifacts, potential geometrical objects do not have actual counterparts. We finally considered infinity, a kind of potentiality which is never actualized, as a model of potentiality for the existence of mathematical entities. But Aristotle, it emerged, does not regard such an unactualizable potentiality as something existent. Thus, insofar as there is no actuality of geometrical objects, they cannot count as real entities in Aristotle's ontological inventory.

What other grounds might there be for thinking mathematical objects existent things for Aristotle? Finally, we examined Hintikka's view that mathematical objects accede to a form of existence in their mental actualization. On the basis of Aristotle's claim of the identity between the thinking intellect and objects of thinking, Hintikka argues that mental actualization is hardly different from physical actualization. But even if we accept that being in actuality in the intellect represents another mode of existence—i.e., an intentional existence in contrast with natural existence, we saw, however, that only forms of externally existing things can be actualized in the intellect in Aristotelian sense. The alternative would blur the distinction between potential things and merely possible or imaginable things, which is utterly crucial to Aristotle's scientific realism. Thus, insofar as the objects of mathematics are not externally actualized, they cannot be actualized in the intellect, either. And if there is no actuality of mathematical objects, they cannot exist in Aristotle's ontology.

Aristotle's philosophy of mathematics grows out of his desire to show that it is

[463] This is also true of the remote matter of primary substances

not necessary to posit Platonic entities in order to account for mathematical truth. Given Aristotle's own theory of truth and scientific realism, though, to show that mathematic is true, mathematical objects must be something existent in his metaphysics. Nevertheless, we saw that neither Aristotle himself nor any commentator was capable of demonstrating this. In this sense, Aristotle's metaphysics cannot provide a model for mathematics; to the extent that Aristotle looked to his metaphysics to explain or secure the truth of mathematics, his project was unsuccessful.

But that does not make Aristotle's philosophy of mathematics undeserving of our attention. His philosophy of mathematics can be said to fail only in the sense that it does not square with his realistic view of sciences and theory of truth. His view is still worth looking at in its own right. Aristotle's failure at least demonstrates how difficult it is for anti-platonists to maintain mathematical realism. If there is no room for mathematical objects in Aristotle's ontology, it can hardly be expected that other forms of anti-platonism will be able to accommodate mathematical realism either; Aristotle's ontology, a form of realism, is far more spacious than most other versions of anti-platonism. More positively, however, we can see in Aristotle's theory of mathematics an anticipation of mathematical fictionalism and of mathematical constructivism in his emphasis on the constructive function of our intellect in the practice of mathematics. Thus, to the extent that mathematical fictionalism can be regarded as a plausible philosophical view of mathematics, to some extent in spite of himself, Aristotle's philosophy of mathematics is vindicated.

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